Lesson 4: Diagonalization

October 5, 2014
The rough idea: From the preceding lesson, we know that a matrix $A$ represents a linear mapping in a certain basis. On the other hand, the matrix associated with a linear mapping changes when we change the basis. So, maybe there exists some basis where the matrix is “specially nice”...
Definition

Let $f : V \rightarrow V$ be an endomorphism. We say that $\lambda \in \mathbb{R}$ (or $\mathbb{C}$) is an eigenvalue of $f$ if there exists $\vec{v} \in V$, $\vec{v} \neq \vec{0}$, such that $f(\vec{v}) = \lambda \vec{v}$. Furthermore, in that case we say that $\vec{v}$ is an eigenvector associated with $\lambda$.

Examples: Khan Academy (click)
How do we compute the eigenvalues: (Khan Academy (click))

Properties:

(1) $p(\lambda) = |A - \lambda I|$ is called the **characteristic polynomial** of the matrix $A$. Its degree is $\text{dim}(V)$.

(2) It is usual to refer to “the eigenvalues of the matrix” (values such that $A \cdot \vec{v} = \lambda \vec{v}$), instead of the linear mapping.
How do we compute the eigenvalues: (Khan Academy (click))

Properties:

(3) If $\lambda_i$ is an eigenvalue, then it is a root of $p(\lambda)$, and therefore

$$p(\lambda) = (\lambda - \lambda_i)^{n_i} \cdots$$

The number $n_i$ is called the algebraic multiplicity of $\lambda_i$.

(4) When we consider vector spaces over $\mathbb{R}$, the eigenvalues can be either real or complex.
How do we compute the eigenvalues: (Khan Academy (click))

Properties:

(5) If $\lambda$ is an eigenvalue of $A$ and $A \cdot \vec{v} = \lambda \vec{v}$, we say that $\vec{v}$ is an eigenvector associated with $\lambda$. 
Proposition

The set of all the eigenvectors associated with a same eigenvalue \( \lambda \) of a matrix \( A \), is a vector subspace.

Proof. Khan Academy (click)

Definition

For each eigenvalue \( \lambda_i \), the set of eigenvectors associated with it is called the eigenspace of \( \lambda_i \). We represent it by \( L_{\lambda_i} \). From the above result, it is a vector subspace, and its dimension is called the geometric multiplicity of \( \lambda_i \).
Observations:

1. $L_{\lambda_i}$ is the solution of $(A - \lambda_i I) \cdot \vec{v} = \vec{0}$.
2. $\dim(L_{\lambda_i}) = n - \text{rank}(A - \lambda_i I)$, where $n$ is the order of $A$.
3. Denoting the algebraic multiplicity of $\lambda_i$ by $n_i$, it holds that

   $$1 \leq \dim(L_{\lambda_i}) \leq n_i$$

Example: Khan Academy (click)
Theorem

Let $f : V \rightarrow V$ be a linear mapping with $p$ different eigenvalues $\lambda_1, \ldots, \lambda_p$. Then the eigenvectors $\vec{v}_1, \ldots, \vec{v}_p$ associated with them are linearly independent.
Introduction: every square matrix $A$ of order $n$ is the matrix associated with some endomorphism $f : V_n \to V_n$ in some basis, i.e. $A = M(f; B, B)$. Furthermore, if we change the basis $B \to B'$, then the matrix changes according to $A' = P^{-1} \cdot A \cdot P$. The question is: given $A$, is there any basis where the expression of the corresponding endomorphism is diagonal? In other words, given $A$, is it similar to any diagonal matrix?

Definition

Let $A$ be a square matrix, and let $f$ be the endomorphism that it represents. We say that $A$ (or $f$) is diagonalizable if there exists some basis such that the matrix associated with $f$ in that basis is diagonal (equivalently, if it is similar to some diagonal matrix).
Diagonalization of a square matrix

**Theorem**

An endomorphism \( f : V_n \to V_n \) is diagonalizable if and only if there exists a basis of \( V_n \) consisting of eigenvectors.

**Proof:** Khan Academy (click)
Diagonalization of a square matrix

**Theorem**

Let $V$ be a vector space over $\mathbb{R}$ of dimension $n$, and let $f : V_n \rightarrow V_n$ be an endomorphism. Then $f$ is diagonalizable (over the reals) if and only if the following two conditions hold:

(i) The total number of real eigenvalues, counting multiplicities, is $n$.

(ii) The geometric multiplicity of each eigenvalue equals its algebraic multiplicity.

Example and observations: Khan Academy (click)