Extensions of discrete classical orthogonal polynomials beyond the orthogonality

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Motivation

1. The Classical orthogonal polynomials
2. The Schemes

The discrete case

1. The Hahn polynomials
2. The other $\Delta$-families
3. Limit relations between hypergeometric orthogonal polynomials
4. Little appendix

The $q$ case

1. The $q$-families
2. The $q$-Hahn polynomials
Basic properties

• Let \((P_n)\) be a polynomial sequence and \(u\) be a linear functional.

• Property of orthogonality

\[
\langle u, P_n P_m \rangle = d_n^2 \delta_{n,m}.
\]

• Distributional equation:

\[
D(\phi u) = \psi u, \quad \deg \psi \geq 1,
\]

where

\[
D = \frac{d}{dx}, \text{ or } D = \nabla, \text{ or } D = \frac{\nabla}{\nabla x(s + 1/2)}.
\]

• Three-term recurrence relation:

\[
x P_n(x) = \alpha_n P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n+1}(x).
\]
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The families

- **Continuous Classical OP:** Jacobi, Hermite, Laguerre and Bessel.
- **$\Delta$-Classical OP:** Hahn, Racah, Meixner, Krawtchouk, Charlier, etc.
- **$q$-Classical OP:** Askey Wilson, $q$-Racah, $q$-Hahn, Continuous $q$-Hahn, Big $q$-Jacobi, $q$-Hermite, $q$-Laguerre, Al-Salam-Chihara, Stieltjes-Wigert, etc.
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Basic properties

Hypergeometric series: \((n = 1, 2, \ldots, N)\)

\[
h_n^{\alpha,\beta}(x; N) = \frac{(-N, \alpha + 1)_n}{(\alpha + \beta + n + 1)_n} \binom{3F_2}{-n, \alpha + \beta + n + 1, -x}{-N, \alpha + 1}.
\]

Property of orthogonality.

\[
\langle u^H, h_n^{\alpha,\beta} h_m^{\alpha,\beta} \rangle = d_n^2 \delta_{n,m}.
\]

Distributional equation:

\[
\Delta (x(\beta + N + 1 - x)u^H) = ((\alpha + 1)N - (\alpha + \beta + 2)x)u^H.
\]

Integral representation with some boundary condition:

\[
\langle u^H, P \rangle = \sum_{x=0}^{N} P(x) \frac{\Gamma(\beta + N + 1 - x)\Gamma(\alpha + 1 + x)}{\Gamma(1 + x)\Gamma(N + 1 - x)}.
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Continuous Hahn polynomials

- Hypergeometric series:
  \[ p_n(x; a, b, c, d) = D_n \, _3F_2 \left( \begin{array}{c} -n, n + a + b + c + d - 1, a + ix \\ a + c, a + d \end{array} \right)_1. \]

- Property of orthogonality: \[ \langle u^{cH}, p_n p_m \rangle = d_n^2 \delta_{n,m}. \]

- Distributional equation: \[ \delta f(x) = f(x + i/2) - f(x - i/2) \]
  \[ \frac{\delta}{\delta x} ((c - ix)(d - ix)) u^{cH} = p_1(x; a, b, c, d) u^{cH}. \]

- Integral representation with some boundary condition:
  \[ \langle u^{cH}, P \rangle = \int_C P(z) \Gamma(a + iz) \Gamma(b + iz) \Gamma(c - iz) \Gamma(d - iz) \, dz, \]
  where \( C \) is a contour on \( \mathbb{C} \) from \( -\infty \) to \( \infty \) which separates the increasing poles from the decreasing ones.
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The limit relation $H\text{-}cH$

- The hypergeometric serie, re-written:

$$h_{n}^{\alpha,\beta}(x; N) = r_{n} \sum_{k=0}^{n} \frac{(-n, \alpha + \beta + n + 1, -x)_{k}(-N + k)_{n-k}}{(\alpha + 1, 1)_{k}},$$

- The limit relation:

$$h_{n}^{\alpha,\beta}(x; N) = \lim_{\varepsilon \to 0} (-i)^{n} p_{n}(ix; 0, \beta + N + \varepsilon + 1, -N - \varepsilon, \alpha + 1).$$
The limit relation $H-cH$

- The hypergeometric serie, re-written:

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- The limit relation:

$$h_{n}^{\alpha,\beta}(x; N) = \lim_{\varepsilon \to 0} (-i)^{n} p_{n}(ix; 0, \beta + N + \varepsilon + 1, -N - \varepsilon, \alpha + 1).$$
About the zeros of the extended Hahn polynomials

Figure: Zeros of $h_{15}^{1,1}(x; 5)$ (left) and $h_{15}^{1,15}(x; 5)$ (right)
The factorization

For any integer $k$, $0 \leq k \leq n$,

$$\Delta^k h_{n}^{\alpha,\beta}(x; N) = (n - k + 1)_k h_{n-k}^{\alpha+k,\beta+k}(x; N - k),$$

$$\nabla^k h_{n}^{\alpha,\beta}(x; N) = (n - k + 1)_k h_{n-k}^{\alpha+k,\beta+k}(x - k; N - k).$$

The factorization:

$$h_{n}^{\alpha,\beta}(x; N) = (x - N)_{N+1}(-i)^{n-N-1}\rho_{n-N-1}(ix; N + 1, \beta + N + 1, 1,$$

$$\alpha + 1) = (x - N)_{N+1}(-i)^{n-N-1}\rho_{n-N-1}((x - \frac{N}{2}) i; 1 + \frac{N}{2}, \beta + 1 +$$

$$+ \frac{N}{2}, 1 + \frac{N}{2}, \alpha + 1 + \frac{N}{2}).$$
The factorization

- For any integer $k$, $0 \leq k \leq n$,

\[
\Delta^k h_{n,k}^{\alpha,\beta}(x; N) = (n - k + 1) h_{n-k}^{\alpha+k,\beta+k}(x; N - k),
\]

\[
\nabla^k h_{n,k}^{\alpha,\beta}(x; N) = (n - k + 1) h_{n-k}^{\alpha+k,\beta+k}(x - k; N - k).
\]

- The factorization:

\[
h_{n,k}^{\alpha,\beta}(x; N) = (x - N)_{N+1}(-i)^{n-N-1} p_{n-N-1}(ix; N + 1, \beta + N + 1, 1, \\
\alpha + 1) = (x - N)_{N+1}(-i)^{n-N-1} p_{n-N-1}((x - \frac{N}{2}) i; 1 + \frac{N}{2}, \beta + 1 + \\
+ \frac{N}{2}, 1 + \frac{N}{2}, \alpha + 1 + \frac{N}{2}).
\]
A characterization Theorem for the Hahn polynomials

**Theorem:** Let $N$ be a non-negative integer and $\alpha, \beta \in \mathbb{C}$ such that:

$-\alpha, -\beta \notin \{1, 2, \ldots, N, N + 2, \ldots\}$, and $-\alpha - \beta \notin \{1, 2, \ldots, 2N + 1, 2N + 3, \ldots\}$. Then the family of Hahn polynomials is a OPS with respect to the $\Delta$-Sobolev inner product:

$$(f, g)_S = \sum_{x=0}^{N} f(x)g(x)\rho^{\alpha,\beta}(x; N) + \int_C (\Delta^{N+1}f(z))(\Delta^{N+1}g(z))\omega^{\alpha,\beta}(z; N)dz,$$

where

$$\rho^{\alpha,\beta}(x; N) = \frac{\Gamma(\beta + N + 1 - x)\Gamma(\alpha + x + 1)}{\Gamma(N + 1 - x)\Gamma(x + 1)},$$

$$\omega^{\alpha,\beta}(z; N) = \Gamma(-z)\Gamma(\beta + N + 1 - z)\Gamma(1 + z)\Gamma(\alpha + N + 2 + z),$$

and $C$ is a complex contour from $-\infty i$ to $\infty i$ which separates the poles of the functions $\Gamma(-z)\Gamma(\beta + N + 1 - z)$ and $\Gamma(1 + z)\Gamma(\alpha + N + 2 + z)$. 
Wilson $\rightarrow$ Racah

- **Factorization:** If $\alpha + 1 = -N$ we get

  \[
  R_n(\lambda(x); \alpha, \beta, \gamma, \delta) = R_{N+1}(\lambda(x); -N - 1, \beta, \gamma, \delta)(-1)^{n-N-1} \\
  \times W_{n-N-1} \left( \left( i \left( x + \frac{\gamma+\delta+1}{2} \right) \right)^2 ; N + \frac{\gamma+\delta+3}{2}, -\frac{\gamma-\delta+1}{2}, \beta + \frac{-\gamma+\delta+1}{2}, \frac{\gamma-\delta+1}{2} \right).
  \]

- **The $\Delta$-Sobolev orthogonality:**

  \[
  \langle p, q \rangle_S = \langle p, q \rangle_d + \left\langle \left( \frac{\Delta}{\Delta \lambda} \right)^{N+1} p, \left( \frac{\Delta}{\Delta \lambda} \right)^{N+1} q \right\rangle_C,
  \]

  with

  \[
  \langle p, q \rangle_d = \sum_{x=0}^{N} p(x)q(x) \frac{(\alpha + 1, \beta + \delta + 1, \gamma + 1, \gamma + \delta + 1, (\gamma + \delta + 3)/2)_x}{(-\alpha + \gamma + \delta + 1, -\beta + \gamma + 1, (\gamma + \delta + 1)/2, \delta + 1, 1)_x},
  \]

  \[
  \langle p, q \rangle_C = \int_C p(z^2)q(z^2)\nu(zi + i + i(\gamma + \delta + N)/2)\nu(-(zi + i + i(\gamma + \delta + N)/2))dz.
  \]
Wilson → Racah

- **Factorization:** If $\alpha + 1 = -N$ we get
  
  \[ R_n(\lambda(x); \alpha, \beta, \gamma, \delta) = R_{N+1}(\lambda(x); -N - 1, \beta, \gamma, \delta)(-1)^{n-N-1} \times W_{n-N-1} \left( \left( i \left( x + \frac{\gamma+\delta+1}{2} \right) \right)^2 ; N + \frac{\gamma+\delta+3}{2}, \frac{-\gamma-\delta+1}{2}, \beta + \frac{-\gamma+\delta+1}{2}, \frac{\gamma-\delta+1}{2} \right). \]

- **The $\Delta$-Sobolev orthogonality:**
  
  \[ \langle p, q \rangle_S = \langle p, q \rangle_d + \left\langle (\Delta/\Delta \lambda)^{N+1} p, (\Delta/\Delta \lambda)^{N+1} q \right\rangle_c, \]

  with

  \[ \langle p, q \rangle_d = \sum_{x=0}^{N} p(x)q(x) \frac{(\alpha + 1, \beta + \delta + 1, \gamma + 1, \gamma + \delta + 1, (\gamma + \delta + 3)/2)_x}{(-\alpha + \gamma + \delta + 1, -\beta + \gamma + 1, (\gamma + \delta + 1)/2, \delta + 1, 1)_x}, \]

  \[ \langle p, q \rangle_c = \int_C p(z^2)q(z^2)\nu(zi + i + i(\gamma + \delta + N)/2)\nu(-(zi + i + i(\gamma + \delta + N)/2))dz. \]
The others

We get analogous results in the following cases:

- Continuous Dual Hahn polynomials $\rightarrow$ Dual Hahn polynomials.
- Meixner $\rightarrow$ Krawtchouk.
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We get analogous results in the following cases:

- Continuous Dual Hahn polynomials $\rightarrow$ Dual Hahn polynomials.
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The limits relations between the families

- **Racah → Hahn.**

\[
\lim_\delta \to \infty R_n(\lambda(x); -N - 1, \beta + \gamma + N + 1, \gamma, \delta) = h_{n,\beta}^{\gamma}(x; N).
\]

- **Racah → Dual Hahn.**

\[
\lim_\beta \to \infty R_n(\lambda(x); -N - 1, \beta, \gamma, \delta) = R_n(\lambda(x); \gamma, \delta, N).
\]

- **Hahn → Krawtchouk.**

\[
\lim_{t \to \infty} h_n^{(1-p)t,p^t}(x; N) = K_n(x; p, N).
\]

- **Dual Hahn → Krawtchouk.**

\[
\lim_{t \to \infty} R_n(\lambda(x); pt, (1-p)t, N) = K_n(x; p, N).
\]
The limits relations between the families

- **Racah → Hahn.**
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  \lim_{\delta \to \infty} R_n(\lambda(x); -N - 1, \beta + \gamma + N + 1, \gamma, \delta) = h_{n,\beta}(x; N).
  \]

- **Racah → Dual Hahn.**
  \[
  \lim_{\beta \to \infty} R_n(\lambda(x); -N - 1, \beta, \gamma, \delta) = R_n(\lambda(x); \gamma, \delta, N).
  \]

- **Hahn → Krawtchouk.**
  \[
  \lim_{t \to \infty} h_n^{(1-p)t, pt}(x; N) = K_n(x; p, N).
  \]

- **Dual Hahn → Krawtchouk.**
  \[
  \lim_{t \to \infty} R_n(\lambda(x); pt, (1-p)t, N) = K_n(x; p, N).
  \]
The limits relations between the families

- **Racah → Hahn.**
  \[
  \lim_{\delta \to \infty} R_n(\lambda(x); -N - 1, \beta + \gamma + N + 1, \gamma, \delta) = h^{\gamma,\beta}_n(x; N).
  \]

- **Racah → Dual Hahn.**
  \[
  \lim_{\beta \to \infty} R_n(\lambda(x); -N - 1, \beta, \gamma, \delta) = R_n(\lambda(x); \gamma, \delta, N).
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  \]
The limits relations between the families

- **Racah → Hahn.**
  $$\lim_{\delta \to \infty} R_n(\lambda(x); -N - 1, \beta + \gamma + N + 1, \gamma, \delta) = h_{n,\beta}^\gamma(x; N).$$

- **Racah → Dual Hahn.**
  $$\lim_{\beta \to \infty} R_n(\lambda(x); -N - 1, \beta, \gamma, \delta) = R_n(\lambda(x); \gamma, \delta, N).$$

- **Hahn → Krawtchouk.**
  $$\lim_{t \to \infty} h_n^{(1-p)t, pt}(x; N) = K_n(x; p, N).$$

- **Dual Hahn → Krawtchouk.**
  $$\lim_{t \to \infty} R_n(\lambda(x); pt, (1-p)t, N) = K_n(x; p, N).$$
Orthogonality relations for Meixner polynomials with general parameter

The Meixner polynomials and continuous Hahn polynomials are related through the following limit relation:

$$\lim_{|t| \to \infty} (-i)^n p_n(ix; 0, -t/c, t, \beta) = M_n(x; \beta, c), \quad n = 0, 1, 2, \ldots.$$ 

**Proposition**: For any $\beta, c \in \mathbb{C}$, $c \notin [0, \infty)$ and $-\beta \notin \mathbb{N}$, the following property of orthogonality for the Meixner polynomials fulfills:

$$\int_C M_n(z; c, \beta)z^m \Gamma(-z)\Gamma(\beta+z)(-c)^z dz = 0, \quad 0 \leq m < n, \quad n = 0, 1, 2, \ldots,$$

where $C$ is a complex contour from $-\infty i$ to $\infty i$ separating the increasing poles $\{0, 1, 2, \ldots \}$ from the decreasing poles $\{-\beta, -\beta - 1, -\beta - 2, \ldots \}$.
The examples considered (under construction)

$q$-Hahn,
$q$-Racah,
dual $q$-Hahn,
quantum $q$-Krawtchouk,
$q$-Krawtchouk,
affine $q$-Krawtchouk, and
dual $q$-Krawtchouk polynomials.
Basic properties

- Basic hypergeometric function: \( n = 1, 2, \ldots, N \)

\[
h_n^{\alpha,\beta}(x; N; q) = \frac{(q^{-N}, q^{\alpha+1}; q)_n}{(q^{\alpha+\beta+n+1}; q)_n} \quad \begin{pmatrix} q^{-n}, q^{\alpha+\beta+n+1}, x \n q^{-N}, q^{\alpha+1} \end{pmatrix}.
\]

- Property of orthogonality.

\[
\langle u^{qH}, h_n^{\alpha,\beta} h_m^{\alpha,\beta} \rangle = d_n^2 \delta_{n,m}.
\]

- Distributional equation:

\[
\mathcal{D}_q(q^{\alpha}(q^{\beta+1}+x - q^{-N})u^{qH}) = h_1^{\alpha,\beta}(x; N; q)u^{qH}.
\]

- Integral representation with some boundary condition:

\[
\langle u^{qH}, P \rangle = \sum_{x=0}^{N} P(x) \frac{(q^{\alpha+1}, q^{-N}; q)_x}{(q, q^{-\beta-N}; q)_x} q^{-(\alpha+\beta)x}.
\]
Basic properties

- Basic hypergeometric function: \((n = 1, 2, \ldots, N)\)

\[
h_n^{\alpha, \beta}(x; N; q) = \frac{(q^{-N}, q^{\alpha+1}; q)_n}{(q^{\alpha+\beta+n+1}; q)_n} \, \, _3\phi_2 \left( \begin{array}{c} q^{-n}, q^{\alpha+\beta+n+1}, x \\ q^{-N}, q^{\alpha+1} \end{array} \right) \bigg| q; q \bigg).
\]

- Property of orthogonality.

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\langle u^{qH}, h_n^{\alpha, \beta} h_m^{\alpha, \beta} \rangle = d_n^2 \delta_{n,m}.
\]

- Distributional equation:

\[
\mathcal{D}_q(q^\alpha (q^{\beta+1} + x - q^{-N}) u^{qH}) = h_1^{\alpha, \beta}(x; N; q) u^{qH}.
\]

- Integral representation with some boundary condition:

\[
\langle u^{qH}, P \rangle = \sum_{x=0}^N P(x) \frac{(q^{\alpha+1}, q^{-N}; q)_x}{(q, q^{-\beta-N}; q)_x} q^{-(\alpha+\beta)x}.
\]
Basic properties

- Basic hypergeometric function: \((n = 1, 2, \ldots, N)\)

\[ h_{n}^{\alpha, \beta}(x; N; q) = \frac{(q^{-N}, q^{\alpha+1}; q)_{n}}{(q^{\alpha+\beta+n+1}; q)_{n}} \, {}_{3}\phi_{2} \left( \begin{array}{c} q^{-n}, q^{\alpha+\beta+n+1}, x \\ q^{-N}, q^{\alpha+1} \end{array} \right| q; q \right). \]

- Property of orthogonality.

\[ \langle u^{qH}, h_{n}^{\alpha, \beta} h_{m}^{\alpha, \beta} \rangle = d_{n}^{2} \delta_{n,m}. \]

- Distributional equation:

\[ D_{q}(q^{\alpha}(q^{\beta+1+x} - q^{-N})u^{qH}) = h_{1}^{\alpha, \beta}(x; N; q)u^{qH}. \]

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\]
The big $q$-Jacobi

- Basic Hypergeometric series:
  \[ P_n(x; a, b, c; q) = \frac{(q^{a+1}, q^{c+1}; q)_n}{(q^{a+b+n+1}; q)_n} \frac{\phi_2}{3}(q^{-n}, q^{a+b+n+1}; x \mid q; q). \]

- Property of orthogonality: \( \langle u^{BqJ}, P_n P_m \rangle = d_n^2 \delta_{n,m}. \)

- Distributional equation:
  \[ \frac{\Delta}{\Delta x} ((x - aq)(x - cq)u^{BqJ}) = P_1(x; a, b, c; q)u^{BqJ}. \]

- Integral representation with some boundary condition:
  \[ \langle u^{BqJ}, P \rangle = \int_{cq}^{aq} P(x) \frac{(q^{-a}x, q^{-c}x; q)_{\infty}}{(x, q^{b-c}x; q)_{\infty}} d_q x. \]
The big $q$-Jacobi

- **Basic Hypergeometric series:**
  $$P_n(x; a, b, c; q) = \frac{(q^{a+1}, q^{c+1}; q)_n}{(q^{a+b+n+1}; q)_n} {_3\phi_2} \left( \begin{array}{c} q^{-n}, q^{a+b+n+1}, x \\ q^{a+1}, q^{c+1} \end{array} \right| q; q \right).$$

- **Property of orthogonality:** $\langle u^{BqJ}, P_n P_m \rangle = d^2_n \delta_{n,m}$.

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The big $q$-Jacobi

- Basic Hypergeometric series:

$$P_n(x; a, b, c; q) = \frac{(q^{a+1}, q^{c+1}; q)_n}{(q^{a+b+n+1}; q)_n} 3\phi_2 \left( \begin{array}{c} q^{-n}, q^{a+b+n+1}, x \\ q^{a+1}, q^{c+1} \end{array} \right| q; q \right).$$

- Property of orthogonality: $\langle u^{BqJ}, P_n P_m \rangle = d_n^2 \delta_{n,m}$.

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- Property of orthogonality: $\langle u^{BqJ}, P_n P_m \rangle = d_n^2 \delta_{n,m}$.

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Roberto Costas

Extensions of $\Delta$-COP beyond the orthogonality
The limit relation $BqJ-qH$

- The Basic hypergeometric series, re-written:
  \[
  h_n^{\alpha, \beta}(x; N; q) = \frac{(q^{\alpha+1}; q)_n}{(q^{\alpha+\beta+n+1}; q)_n} \sum_{k=0}^{n} \frac{(q^{-n}, q^{\alpha+\beta+n+1}, q^{-x}; q)_k (q^{-N+k}; q)_{n-k}}{(q^{\alpha+1}, q; q)_k} q^k.
  \]

- The key...the following algebraic relation:
  \[
  \frac{(q^{i+1}; q)_{n-N-1-j}(q^{-n}; q)_{N+1+j}}{(q; q)_{N+1+j}} = \frac{(q; q)_{n-N-1}(q^{-n}; q)_{N+1}(q^{-n+N+1}; q)_j}{(q; q)_{N+1}(q^{N+2}, q; q)_j}.
  \]

- The limit relation:
  \[
  h_n^{\alpha, \beta}(x; N; q) = \lim_{\varepsilon \to 0} P_n(x; \alpha, \beta, -N - 1 + \varepsilon; q).
  \]
The limit relation $BqJ$-$qH$

- The Basic hypergeometric series, re-written:

$$h_{\alpha,\beta}^n(x; N; q) = \frac{(q^{\alpha+1}; q)_n}{(q^{\alpha+\beta+n+1}; q)_n} \sum_{k=0}^{n} \frac{\left(q^{-n}, q^{\alpha+\beta+n+1}, q^{-x}; q\right)_k(q^{-N+k}; q)_{n-k}}{(q^{\alpha+1}, q; q)_k} q^k.$$ 

- The key...the following algebraic relation:

$$\frac{(q^{i+1}; q)_{n-N-1-j}(q^{-n}; q)_{N+1+j}}{(q;q)_{N+1+j}} = \frac{(q; q)_{n-N-1}(q^{-n}; q)_{N+1}(q^{-n+N+1}; q)_{j}}{(q; q)_{N+1}(q^{N+2}, q; q)_j}.$$ 

- The limit relation:

$$h_{\alpha,\beta}^n(x; N; q) = \lim_{\varepsilon \to 0} P_n(x; \alpha, \beta, -N - 1 + \varepsilon; q).$$
The limit relation $BqJ\cdot qH$

- The Basic hypergeometric series, re-written:

$$h_{n}^{\alpha,\beta}(x; N; q) = \frac{(q^{\alpha+1}; q)_{n}}{(q^{\alpha+\beta+n+1}; q)_{n}} \sum_{k=0}^{n} \frac{(q^{-n}, q^{\alpha+\beta+n+1}, q^{-x}; q)_{k}(q^{-N+k}; q)_{n-k}}{(q^{\alpha+1}, q; q)_{k}} q^{k}.$$ 

- The key...the following algebraic relation:

$$\frac{(q^{i+1}; q)_{n-N-1-j}(q^{-n}; q)_{N+1+j}}{(q; q)_{N+1+j}} = \frac{(q; q)_{n-N-1}(q^{-n}; q)_{N+1}(q^{-n+N+1}; q)_{j}}{(q; q)_{N+1}(q^{N+2}, q; q)_{j}}.$$ 

- The limit relation:

$$h_{n}^{\alpha,\beta}(x; N; q) = \lim_{\varepsilon \to 0} P_{n}(x; \alpha, \beta, -N - 1 + \varepsilon; q).$$
About the zeros of the extended $q$-Hahn polynomials

**Figure:** Log of the zeros of $h_{15}^{1,1}(x; 5; 0.5)$ (left) and $h_{15}^{1,1}(x; 5; \exp(0.23I))$ (right)
The factorization

For any integer $k$, $0 \leq k \leq n$,

$$D_{q-1}^k h_{n-k}^{\alpha,\beta} (x; N; q) = (n - k + 1|q^{-1})_k h_{n-k}^{\alpha+k,\beta+k} (x; N - k; q),$$

$$D_q^k h_{n-k}^{\alpha,\beta} (x; N; q) = (n - k + 1|q^{-1})_k h_{n-k}^{\alpha+k,\beta+k} (x - k, N - k; q),$$

The factorization:

$$h_{n}^{\alpha,\beta} (q^{-x}; N; q) = \frac{h_{N+1}^{\alpha,\beta} (q^{-x}; N; q)}{q^{(N+1)(n-N-1)}} P_{n-N-1}(-x+N+1; \alpha+N+1, \beta+N+1, N+1; q).$$
For any integer $k$, $0 \leq k \leq n$,

\[
D_{q-1}^k h_{n}^{\alpha,\beta}(x; N; q) = (n - k + 1|q^{-1})_k h_{n-k}^{\alpha+k,\beta+k}(x; N - k; q),
\]

\[
D_q^k h_{n}^{\alpha,\beta}(x; N; q) = (n - k + 1|q^{-1})_k h_{n-k}^{\alpha+k,\beta+k}(x - k, N - k; q),
\]

The factorization:

\[
h_{n}^{\alpha,\beta}(q^{-x}; N; q) = \frac{h_{N+1}^{\alpha,\beta}(q^{-x}; N; q)}{q^{(N+1)(n-N-1)}} P_{n-N-1}(-x+N+1; \alpha+N+1, \beta+N+1, N+1; q).
\]
A characterization Theorem for the $q$-Hahn polynomials

Theorem: Let $N$ be a non-negative integer and $\alpha, \beta \in \mathbb{C}$ such that: $-\alpha, -\beta \notin \{1, 2, \ldots, N, N + 2, \ldots\}$, and $-\alpha - \beta \notin \{1, 2, \ldots, 2N + 1, 2N + 3, \ldots\}$. Then the family of Hahn polynomials is a OPS with respect to the $\mathcal{D}_{q-1}$-Sobolev inner product:

$$(f, g)_s = \sum_{x=0}^{N} f(x)g(x)\rho^{\alpha, \beta}(x; N; q) + \int_{\mathcal{C}} (\mathcal{D}^{N+1}_{q-1} f(z))(\mathcal{D}^{N+1}_{q-1} g(z))\omega^{\alpha, \beta}(z; N; q)dz$$

where

$$\rho^{\alpha, \beta}(x; N; q) = \frac{\Gamma_q(\alpha+1+x)\Gamma_q(x-N)}{\Gamma_q(1+x)\Gamma_q(-\beta-N+x)} q^{-(\alpha+\beta)x},$$

and $\mathcal{C}$ is a complex contour from $-\infty i$ to $\infty i$ which separates the certain poles*.
Some references


Some references


Finally...

Thank for your attention!!