An overview of Classical Orthogonal Polynomials

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   - The Classical Hypergeometric Orthogonal Polynomials
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3. Some Results
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   - Hypergeometric and basic hypergeometric representations
   - The Connection Problem
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THE BASICS

Classical Orthogonal Polynomials

- Let (P_n) be a polynomial sequence and $\mathbf{u}$ be a functional.
- Property of orthogonality

$$\langle \mathbf{u}, P_n P_m \rangle = d_n^2 \delta_{n,m}.$$

- Distributional equation:

$$\mathscr{D}(\phi \mathbf{u}) = \psi \mathbf{u}, \quad \deg \psi \geq 1, \ \deg \phi \leq 2.$$

- Three-term recurrence relation:

$$x P_n(x) = \alpha_n P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n+1}(x).$$

- The weight function $d\mu(z) = \omega(z)\, dz$

$$\langle \mathbf{u}, P \rangle = \int_\Gamma P(z)\, d\mu(z), \qquad \Gamma \subset \mathbb{C}, .$$

1. Continuous classical orthogonal polynomials
   - $\dfrac{d}{dx}(\phi(x)\omega(x)) = \psi(x)\omega(x)$,
2. Δ-classical orthogonal polynomials
   - $\nabla(\phi(x)\omega(x)) = \psi(x)\omega(x)$,
   - $\Delta f(x) = f(x+1) - f(x)$, $\nabla f(x) = f(x) - f(x-1)$,
3. q-Hahn classical orthogonal polynomials
   - $\mathscr{D}_{1/q}(\phi(x)\omega(x)) = \psi(x)\omega(x)$,
   - $\mathscr{D}_q f(x) = \frac{f(qx)-f(x)}{x(q-1)}$, $x \neq 0$, $\mathscr{D}_q f(0) = f'(0)$,
   - $x(s) = c_1 q^s + c_2$.

- Continuous Classical OP: Jacobi, Hermite, Laguerre and Bessel.
- Δ-Classical OP: Hahn, Racah, Meixner, Krawtchouk, Charlier, etc.
- q-Classical OP: Askey Wilson, q-Racah, q-Hahn, Continuous q-Hahn, Big q-Jacobi, q-Hermite, q-Laguerre, Al-Salam-Chihara, Stieltjes-Wigert, etc.

Some families

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Let $(p_n)_{n\in\mathbb{N}_0}$ generated by the TTRR

$$xp_n(x) = p_{n+1}(x) + \beta_n p_n(x) + \gamma_n p_{n-1}(x).$$

Favard's theorem

If $\gamma_n \neq 0 \ \forall n \in \mathbb{N}$ then there exists a moments functional $\mathscr{L}_0 : \mathbb{P}[x] \to \mathbb{C}$ so that

$$\mathscr{L}_0(p_n p_m) = r_n \delta_{n,m}$$

with r_n a non-vanishing normalization factor.

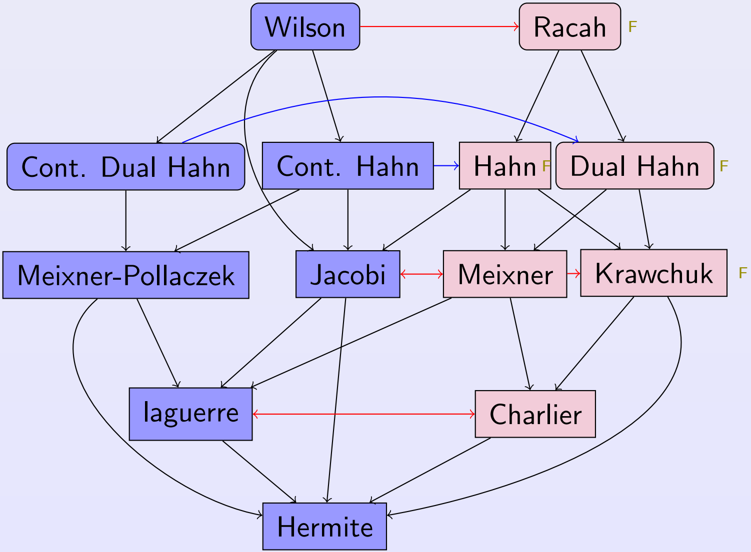
Theorem

If there exists N so that $\gamma_N = 0$, then (p_n) is a MOPS with respect to

$$\langle f, g\rangle = \mathscr{L}_0(fg) + \sum_{j\in\mathscr{A}} \mathscr{L}_1(\mathscr{T}^{(N)}(f)\mathscr{T}^{(N)}(g)).$$

THE RELEVANT FAMILIES

The Classical Hypergeometric Orthogonal Polynomials



The scheme is too big to put it on here, let's go outside to see it ;)

SOME RESULTS

Characterization Theorems. The continuous version

Let (P_n) be an OPS with respect to ω. The following statements are equivalent:

1. P_n is classical, i.e. $(\phi(x)\omega(x))' = \psi(x)\omega(x)$.
2. (P'_{n+1}) is a OPS.
3. $(P^{(k)}_{n+k})$ is a OPS for any integer k.
4. (First structure relation)

$$\phi(x)P'_n(x) = \widehat{\alpha}_n P_{n+1}(x) + \widehat{\beta}_n P_n(x) + \widehat{\gamma}_n P_{n-1}(x).$$

5. (Second structure relation)

$$P_n(x) = \widetilde{\alpha}_n P'_{n+1}(x) + \widetilde{\beta}_n P'_n(x) + \widetilde{\gamma}_n P'_{n-1}(x).$$

6. (Eigenfunctions of SODE)

$$\phi(x)P''(x) + \psi(x)P'(x) + \lambda P(x) = 0.$$

Characterization Theorem (cont.)

Let (P_n) be an OPS with respect to ω. The following statements are equivalent:

1. P_n is classical, i.e. $(\phi(x)\omega(x))' = \psi(x)\omega(x)$.
2. The Rodrigues Formula for P_n

$$P_n(x) = \frac{B_n}{\omega(x)} \frac{d^n}{dx^n} (\phi^n(x)\omega(x)), \qquad B_n \neq 0.$$

3. $\phi(x)(P_nP_{n-1})'(x) =$
$g_nP_n^2(x) - (\psi(x) - \phi'(x))P_n(x)P_{n-1}(x) + h_nP_{n-1}^2(x)$

Hypergeometric and basic hypergeometric representations

The continuous and discrete COP can be written in terms of

$$ {}_rF_s\left(\begin{array}{c} a_1,\ a_2,\ \ldots, a_r \\ b_1,\ b_2,\ \ldots, b_s \end{array} \middle| z\right) = \sum_{k\geq 0} \frac{(a_1)_k (a_2)_k \ldots (a_r)_k}{(b_1)_k (b_2)_k \ldots (b_s)_k} \frac{z^k}{k!}. $$

The q-discrete COP can be written in terms of

$$ {}_r\varphi_s\left(\begin{array}{c} a_1,\ \ldots, a_r \\ b_1,\ \ldots, b_s \end{array} \middle| z\right) = \sum_{k\geq 0} \frac{(a_1;q)_k \ldots (a_r;q)_k}{(b_1;q)_k \ldots (b_s;q)_k} \left((-1)^k q^{\binom{k}{2}}\right)^{1+s-r} \frac{z^k}{(q;q)_k}. $$

$$ (a)_k = a(a+1)\cdots(a+k-1) $$

$$ (a;q)_k = (1-a)(1-aq)\cdots(1-aq^{k-1}) $$

The Connection Problem

The connection problem is the problem of finding the coefficients $c_{k;n}$ in the expansion of P_n in terms of another sequence of polynomials R_k, i.e.

$$P_n(x) = \sum_{k=0}^{n} c_{k;n} R_k(x).$$

We are interested into obtaining such coefficients for Classical orthogonal polynomials in a enough 'general' context.

The example. Big q-Jacobi polynomials

`Again let's go to File 2 :D`

Some References

- (with J.F. Sánchez-Lara) Extensions of discrete classical orthogonal polynomials beyond the orthogonality. *J. Comput. Appl. Math. 225 (2009), no. 2, 440–451*
- (with F. Marcellán) q-Classical orthogonal polynomial: A general difference calculus approach. *Acta Appl. Math. 111 (2010), no. 1, 107–128*
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THANK YOU

FOR YOUR ATTENTION !!