Old and new results on Sobolev and SemiClassical Orthogonal Polynomials

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   - Semiclassical Sobolev Orthogonal Polynomials
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Let \((P_n)\) be a polynomial sequence and \(u\) be a functional.

Property of orthogonality

\[ \langle u, P_n P_m \rangle = d_n^2 \delta_{n,m}. \]

Distributional equation:

\[ \mathcal{D}(\phi u) = \psi u, \quad \deg \psi \geq 1, \ deg \phi \leq 2. \]

Three-term recurrence relation:

\[ xP_n(x) = \alpha_n P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n+1}(x). \]

COP: Jacobi, Hermite, Laguerre, Bessel.

Discrete-COP: Hahn, Racah, Meixner, Krawtchouk, Charlier, Askey-Wilson, \(q\)-Racah, \(q\)-Hahn, Al Salam Carlitz I, II, etc.
Favard’s theorem

Let \((p_n)_{n \in \mathbb{N}_0}\) generated by the TTRR

\[ xp_n(x) = p_{n+1}(x) + \beta_n p_n(x) + \gamma_n p_{n-1}(x). \]

If \(\gamma_n \neq 0 \quad \forall n \in \mathbb{N}\) then there exists a moments functional
\(\mathcal{L}_0 : \mathbb{P}[x] \to \mathbb{C}\) so that

\[ \mathcal{L}_0(p_n p_m) = r_n \delta_{n,m} \]

with \(r_n\) a non-vanishing normalization factor.
Degenerate version of Favard’s theorem. Some history

- K. H. Kwon and L. L. Littlejohn, $(L_n^{(-k)})$ orthogonal w.r.t.

\[
\langle f, g \rangle = (f(0), f'(0), \ldots, f^{(k-1)}(0))A(g(0), g'(0), \ldots, g^{(k-1)}(0))^T + \int_0^\infty f^{(k)}(x)g^{(k)}(x)e^{-x}dx.
\]

- The Jacobi polynomials $(P_n^{(-1,-1)})$ orthogonal w.r.t.

\[
(f, g)_1 = d_1 f(1)g(1) + d_2 f(-1)g(-1) + \int_{-1}^1 f'(x)g'(x)dx.
\]

- These examples suggest that COP with non-classical parameters can be provided with a orthogonality of Sobolev-type.

- Furthermore F. Marcellan and J.J. Moreno-Balcázar pointed out that a Sobolev-Askey tableau should be established.
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- These examples suggest that COP with non-classical parameters can be provided with a orthogonality of Sobolev-type.

- Furthermore F. Marcellán and J.J. Moreno-Balcázar pointed out that a Sobolev-Askey tableau should be established.
Degenerate version of Favard’s theorem. The preliminaries

- Let \( \mathcal{T}_1 : \mathbb{P}[x] \rightarrow \mathbb{P}[x] \) be a linear operator such that
  - \( \deg \mathcal{T}_1(p) = \deg p - 1 \)
  - The monic polynomials sequence \((p_{n,1})\) defined by
    \[
    p_{n,1} := \text{const.} \mathcal{T}_1(p_{n+1}),
    \]
    fulfill the TTRR
    \[
    xp_{n,1}(x) = p_{n+1,1}(x) + \beta_{n,1} p_{n,1}(x) + \gamma_{n,1} p_{n-1,1}(x)
    \]
    so that there exists \( \lambda : \{ \gamma_{n,1} = 0 \} \rightarrow \{ \gamma_n = 0 \} \) strictly increasing with \( \lambda(n) > n \).

Remark

\((p_{n,1})\) is orthogonal with respect to some moments functional \( \mathcal{L}_1 \).
Degenerate version of Favard’s theorem. The preliminaries

Let \( T_1 : \mathbb{P}[x] \to \mathbb{P}[x] \) be a linear operator such that
- \( \deg T_1(p) = \deg p - 1 \)
- The **monic** polynomials sequence \((p_{n,1})\) defined by

\[
p_{n,1} := \text{const.} \cdot T_1(p_{n+1}),
\]

fulfill the TTRR

\[
x p_{n,1}(x) = p_{n+1,1}(x) + \beta_{n,1} p_{n,1}(x) + \gamma_{n,1} p_{n-1,1}(x)
\]

so that there exists \( \lambda : \{\gamma_{n,1} = 0\} \to \{\gamma_n = 0\} \) strictly increasing with \( \lambda(n) > n \).

**Remark**

\((p_{n,1})\) is orthogonal with respect to some moments functional \( \mathcal{L}_1 \).
The iterative process

1. \( p_{n,k} := \text{const.} \mathcal{T}_k(p_{n+1,k-1}) = \cdots = \text{const.} \mathcal{T}^{(k)}(p_{n+k}) \).
2. \( xp_{n,k}(x) = p_{n+1,k}(x) + \beta_{n,k} p_{n,k}(x) + \gamma_{n,k} p_{n-1,k}(x) \)
3. \( \mathcal{L}_k(p_m,k p_{n,k}) = 0 \) for \( n \neq m \).
4. The first \( n \) such that \( \gamma_{n,k} = 0 \) (if it exists) verifies \( n < N - k \).

Theorem:
Suppose that only \( \gamma_N = 0 \), then \( (p_n) \) is a MOPS with respect to

\[
\langle f, g \rangle = \mathcal{L}_0(fg) + \mathcal{L}_N(\mathcal{T}^{(N)}(f)\mathcal{T}^{(N)}(g)).
\]

Notice \( \gamma_{n,N} \neq 0 \) for all \( n \in \mathbb{N} \).
The iterative process

1. \( p_{n,k} := \text{const.} \mathcal{T}_k(p_{n+1,k-1}) = \cdots = \text{const.} \mathcal{T}^{(k)}(p_{n+k}) \).
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Suppose that only \( \gamma_N = 0 \), then \((p_n)\) is a MOPS with respect to

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\]

Notice \( \gamma_{n,N} \neq 0 \) for all \( n \in \mathbb{N} \).
Degenerate version of Favard’s theorem

Corollary

If $\Lambda = \{n : \gamma_n = 0\}$, then $(p_n)$ is a MOPS with respect to

$$\langle f, g \rangle = \mathcal{L}_0(fg) + \sum_{j \in \mathcal{A}} \mathcal{L}_j(T^{(j)}(f)T^{(j)}(g)),$$

being $\mathcal{A} = \{N_0, N_1, \ldots\}$ with $N_{j+1} = N_j + \min\{n : \gamma_n, N_j = 0\}$. 
Among all the possible choices the linear operator $T$ can be chosen as the “Associating operator”

$$T(p)(x) = L_0 \left( \frac{p(x) - p(t)}{x - t} \right).$$

2. If $(p_n)$ is classical, then $T$ is the derivative, or a difference operator.
3. And now ... the example.
1 Among all the possible choices the linear operator $T$ can be chosen as The “Associating operator”

$$T(p)(x) = \mathcal{L}_0 \left( \frac{p(x) - p(t)}{x - t} \right).$$

2 If $(p_n)$ is classical, then $T$ is
   - the derivative, or
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3 And now ... the example.
Among all the possible choices the linear operator $\mathcal{T}$ can be chosen as The “Associating operator”

$$\mathcal{T}(p)(x) = \mathcal{L}_0 \left( \frac{p(x) - p(t)}{x - t} \right).$$

If $(p_n)$ is classical, then $\mathcal{T}$ is the derivative, or a difference operator.

And now ... the example.
The monic ones are $p_n(x; a, b, c, d; q) \equiv p_n(x)$

$$p_{n+1}(x) = (x - \beta_n)p_n(x) - \gamma_n p_{n-1}(x),$$

with

$$\frac{\gamma_n}{1 - q^n} = \frac{(1 - abq^{n-1})(1 - acq^{n-1})(1 - adq^{n-1})(1 - bcq^{n-1})(1 - bdq^{n-1})(1 - cdq^{n-1})}{4(1 - abcdq^{2n-3})(1 - abcdq^{2n-2})^2(1 - abcdq^{2n-1})}$$

Case $abcd \in \{q^{-k} : k \in \mathbb{N}_0\}$ are not considered since they are not normal.
They are symmetric with respect to any rearrangement of the parameters $a, b, c, d$.

$$\{n \in \mathbb{N} : \gamma_n = 0\} \neq \emptyset \iff ab, ac, \ldots, cd \in \{q^{-k} : k \in \mathbb{N}_0\}$$

$\iff$ they are $q$-Racah.
Orthogonality of AW polynomials for $|q| < 1$

\[ \int_{C} p_n \left( \frac{z + z^{-1}}{2} \right) p_m \left( \frac{z + z^{-1}}{2} \right) W(z) dz = d_n \delta_{n,m} \]

where

- $W$ is analytic in $\mathbb{C}$ except at the poles $0, aq^k, bq^k, cq^k, dq^k \quad k \in \mathbb{N}_0$ (the convergent poles)

- $(aq)^{-k}, (bq)^{-k}, (cq)^{-k}, (dq)^{-k} \quad k \in \mathbb{N}_0$ (the divergent poles)

- $C$ is the unit circle deformed to separate the convergent form the divergent poles.
The 3 key cases

- **Case I:** \( a^2 = q^{-N+1} \) and
  \[ b^2, c^2, d^2, ab, ac, ad, bc, bd, cd \not\in \{q^{-k} : k \in \mathbb{N}_0\} \]

- **Case II:** \( ab = q^{-N+1} \) and
  \[ a^2, b^2, c^2, d^2, ac, ad, bc, bd, cd \not\in \{q^{-k} : k \in \mathbb{N}_0\} \]

- **Case III:** \( ab = q^{-N+1}, a^2 = q^{-M} \) with \( M \in \{0, 1, \ldots, N - 2\} \) and
  \[ b^2, c^2, d^2, ac, ad, bc, bd, cd \not\in \{q^{-k} : k \in \mathbb{N}_0\} \]
Orthogonality of AW polynomials for $|q| \geq 1$

- $|q| > 1$: By using the identity
  \[ p_n(x; a, b, c, d|q^{-1}) = p_n(x; a^{-1}, b^{-1}, c^{-1}, d^{-1}|q) \]

- $|q| = 1$: If $q = \exp(2M\pi/N \, i)$, then $\gamma_jN = 0, j \in \mathbb{N}$.
  - Spiridonov and Zhedanov found $L_0$
  - For $n > N$
    \[ \mathcal{D}^N p_n(x; a, b, c, d|q) = p_{n-N}((-1)^M x; a, b, c, d|q). \]
    - $L_j(p(\cdot)) = L_0(p((-1)^M \cdot))$
  - For the rest of the values of $q$ the result keeps unknown.
Let \((P_n)\) be a polynomial sequence and \(u\) be a functional.

Property of quasi-orthogonality of order \(\delta\)

\[
\langle u, P_n P_m \rangle = 0 \quad |n - m| > \delta, \quad \exists r \geq \delta : \langle u, P_r P_{r-\delta} \rangle \neq 0.
\]

Distributional equation:

\[
\mathcal{D}(\phi u) = \psi u, \quad \text{deg} \psi \geq 1.
\]

A ’recurrence relation’:

\[
P_n = M_{r-1} Z_{n-r+1} + N_{r-2} Z_{n-r}.
\]

There is not a general classification.
Some definitions

**Admisibility**

The pair of polynomials \((\phi, \psi)\) is an admissible pair if one of the following conditions is satisfied:

- \(\deg \psi \neq \deg \phi - 1\),
- \(\deg \psi = \deg \phi - 1\), with \(a_p + q^{-1}[n]^* b_t \neq 0\), where \(a_p\) and \(b_t\) are the leading coefficients of \(\psi\) and \(\phi\), respectively.

**Order and class of a linear functional**

\(\sigma := \max\{\deg \phi - 2, \deg \psi - 1\}\). The class of \(u\) is the min. order from among all the adm. pairs.

**The sequence \((\phi_k)\) and \((u_k)\)**

Given a semiclassical functional \(u\) satisfying PE, for \(k \in \mathbb{Z}\) we define the \(u_k\) as: \(u_k = \mathcal{S}^+(\phi_{k-1} u_{k-1})\), \(u_0 = u\), \(\phi_0 = \phi\), where \(\phi_k\) is a multiple of \(\phi_{k-1}\).
Some definitions

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Given a semiclassical functional \(u\) satisfying PE, for \(k \in \mathbb{Z}\) we define the \(u_k\) as: \(u_k = \mathcal{E}^+(\phi_{k-1} u_{k-1})\), \(u_0 = u\), \(\phi_0 = \phi\), where \(\phi_k\) is a multiple of \(\phi_{k-1}\).
Theorem 1

Let \((p_n)\) be a sequence of monic OP w.r.t. \(u\), and \(\phi\) pol. of degree \(t\). The following statements are equivalent:

1. There exist three non-negative integers, \(\sigma\), \(p\), and \(r\), with \(p \geq 1\), \(r \geq \sigma + t + 1\), and \(\sigma = \max\{t - 2, p - 1\}\), s.t.

\[
\xi_{n,\nu} p_{\nu}(z) = \sum_{\nu=n-t}^{n+\sigma} \varsigma_{n,\nu} p_{\nu}^{[1]}(z),
\]

where \(p_{n}^{[1]}(z) := [n + 1]^{-1}(D p_{n+1})(z)\).

2. There exists a polynomial \(\psi\), with \(\deg \psi = p \geq 1\), such that

\[
D(\phi u) = \psi u,
\]

where the pair \((\phi, \psi)\) is an admissible pair.
Let $u$ be a semiclassical functional of order $\sigma$, and let $(p_n)$ be the monic OP sequence w.r.t. $u$.

Consider the $\mathcal{D}$–Sobolev inner product defined by

$$\langle p, r \rangle_S = \langle u, p r \rangle + \lambda \langle u, \mathcal{D} p \mathcal{D} r \rangle, \quad \lambda \geq 0.$$ 

Let $(Q_n^{(\lambda)})$ be the OP sequence associated with the ($\mathcal{D}$–Sobolev) inner product $\langle \cdot, \cdot \rangle_S$ which we call semiclassical Sobolev orthogonal polynomial.

**Proposition**

For $n \geq \sigma - 1 + H^*$,

$$\sum_{\nu = n - \sigma - 1}^{n + \sigma - 1} \xi_{n, \nu} p_{\nu}^{(-1)}(z) = Q_n^{(\lambda)}(z) + \sum_{\nu = n - \sigma - 1 - H^*}^{n - 1 + \sigma - 1} \theta_{n, \nu} Q_{\nu}^{(\lambda)}(z),$$

where $H^* := \max\{t - 1, \sigma - 1\}$.
Let \( u \) be a semiclassical functional of order \( \sigma \), and let \((p_n)\) be the monic OP sequence w.r.t. \( u \).

Consider the \( \mathcal{D} \)-Sobolev inner product defined by
\[
\langle p, r \rangle_S = \langle u, p \, r \rangle + \lambda \langle u, \mathcal{D} p \, \mathcal{D} r \rangle, \quad \lambda \geq 0.
\]

Let \((Q_n^{(\lambda)})\) be the OP sequence associated with the \((\mathcal{D} \)-Sobolev\) inner product \( \langle \cdot, \cdot \rangle_S \) which we call semiclassical Sobolev orthogonal polynomial.

**Proposition**

For \( n \geq \sigma_{-1} + H^* \),
\[
\sum_{\nu = n - \sigma_{-1}}^{n + \sigma_{-1}} \xi_{n, \nu} p_{\nu}^{\{-1\}}(z) = Q_{n + \sigma_{-1}}^{(\lambda)}(z) + \sum_{\nu = n - \sigma_{-1} - H^*}^{n - 1 + \sigma_{-1}} \theta_{n, \nu} Q_{\nu}^{(\lambda)}(z),
\]
where \( H^* := \max\{t_{-1}, \sigma_{-1}\} \).
two identities and a Theorem

Identity 1

Let $\mathcal{J}$ be the linear operator

$$\mathcal{J} := (E - \tilde{\phi}) I + \frac{\lambda}{q} (D^* \tilde{\phi} - \tilde{\psi}) D^* - \lambda (E - \tilde{\phi}) D D^*,$$

where $I$ is the identity operator. Then,

$$\langle (E - \tilde{\phi}) p, r \rangle_S = \langle u, p \mathcal{J} r \rangle, \quad p, r \in \mathbb{P}. $$

Identity 2

$$\langle (\tilde{\psi} - D^* \tilde{\phi}) p, r \rangle_S = \langle D^* u, p \mathcal{J} r \rangle, \quad p, r \in \mathbb{P}. $$

Theorem

$$\langle \mathcal{J} p, r \rangle_S = \langle p, \mathcal{J} r \rangle_S, \quad p, r \in \mathbb{P}. $$
two identities and a Theorem

Identity 1

Let $\mathcal{J}$ be the linear operator

$$\mathcal{J} := (E - \tilde{\phi}) \mathcal{I} + \frac{\lambda}{q} (D^* \tilde{\phi} - \tilde{\psi}) D^* - \lambda (E - \tilde{\phi}) D D^*,$$

where $\mathcal{I}$ is the identity operator. Then,

$$\langle (E - \tilde{\phi}) p, r \rangle_S = \langle u, p \mathcal{J} r \rangle, \quad p, r \in \mathbb{P}.$$

Identity 2

$$\langle (\tilde{\psi} - D^* \tilde{\phi}) p, r \rangle_S = \langle D^* u, p \mathcal{J} r \rangle, \quad p, r \in \mathbb{P}.$$

Theorem

$$\langle \mathcal{J} p, r \rangle_S = \langle p, \mathcal{J} r \rangle_S, \quad p, r \in \mathbb{P}.$$
two identities and a Theorem

Identity 1

Let \( J \) be the linear operator

\[
J := (E - \tilde{\phi}) I + \frac{\lambda}{q} (D^* \tilde{\phi} - \tilde{\psi}) D^* - \lambda (E - \tilde{\phi}) D D^*,
\]

where \( I \) is the identity operator. Then,

\[
\langle (E - \tilde{\phi}) p, r \rangle_S = \langle u, p J r \rangle, \quad p, r \in \mathbb{P}.
\]

Identity 2

\[
\langle (\tilde{\psi} - D^* \tilde{\phi}) p, r \rangle_S = \langle D^* u, p J r \rangle, \quad p, r \in \mathbb{P}.
\]

Theorem

\[
\langle J p, r \rangle_S = \langle p, J r \rangle_S \quad p, r \in \mathbb{P}.
\]
The corollary

Corollary

The following relations hold

\[(\mathcal{E} - \tilde{\phi})(z)p_n(z) = \sum_{\nu = n - H}^{\nu = n + \deg \tilde{\phi}} \mu_{n, \nu} Q_{\nu}^{(\lambda)}(z), \quad n \geq H,\]

\[J Q_n^{(\lambda)}(z) = \sum_{\nu = n - \deg \tilde{\phi}}^{n + H} \vartheta_{n, \nu} p_{\nu}(z), \quad n \geq \deg \tilde{\phi},\]

\[J Q_n^{(\lambda)}(z) = \sum_{\nu = n - H}^{n + H} \varpi_{n, \nu} Q_{\nu}^{(\lambda)}(z), \quad n \geq H,\]

where \(H := \max\{\deg \tilde{\psi} - 1, \deg \tilde{\phi}\}\).
A $q$ example: $q$-Freud type polynomials

The monic $q$–Freud polynomials, $(P_n)$, satisfies the relation

$$(\mathcal{D}P_n)(x(s)) = [n]P_{n-1}(x(s)) + a_nP_{n-3}(x(s)), \quad n \geq 0,$$

where $x(s) = q^s$, with $0 < q < 1$, $\mathcal{D} = \mathcal{D}_q$.

- $\phi(x) = 1$, $t = 0$, and $(P_n)$ is orth. w.r.t. $u^{qF}$ of class $\sigma = 2$.
- $\mathcal{D}(u^{qF}) = \psi u^{qF}$, $\mathcal{D}^*((1 + x(q - 1)\psi)u^{qF}) = q\psi u^{qF}$, $\deg \psi = 3$.
- $u^{qF}$ has the following integral representation:

$$\langle u^{qF}, P \rangle = \int_{-1}^{1} P(x) \frac{1}{((q - 1)K(q)q^{-3}q^{4x}; q^4)_\infty} dq(x),$$
\[ J^{qF} = (1 - (q - 1)K(q)q^{-7}x^4)J + \frac{\lambda}{q}K(q)q^{-6}x^3 D_{1/q} - \lambda(1 - (q - 1)K(q)q^{-7}x^4)D_q D_{1/q}. \]

- \( H = \deg \tilde{\phi} = 4 \) and

\[ (1 - (q - 1)K(q)q^{-7}x^4)P_n(x) = \sum_{\nu=n-4}^{n+4} \mu_{n,\nu}^{qF} Q_{\nu}^{qF}(x), \]

\[ J^{qF} Q_{n}^{qF}(x) = \sum_{\nu=n-4}^{n+4} \phi_{n,\nu}^{qF} P_{\nu}(x), \]

\[ J^{qF} Q_{n}^{qF}(x) = \sum_{\nu=n-4}^{n+4} \omega_{n,\nu}^{qF} Q_{\nu}^{qF}(x). \]
This family, \((S_n)\), was studied by J.C. Medem and it is orthogonal w.r.t. \(w\).

**Distributional equation:**

\[
\mathcal{D}(x^3w) = (-x^2 + 4)w.
\]

\(\mathcal{D} = \mathcal{D}^* = \frac{d}{dx}, \ t = 3, \ p = 2, \ so \ \sigma = 1; \ with \ initial \ condition \ (w)_1 = \langle w, x \rangle = 0.\)

\(w\) is 1–singular.

\(w\) is the symmetrized of \(b^{(-\frac{5}{2})}\).

\(w\) has the following integral representation:

\[
\langle w, P \rangle = \frac{1}{2\pi i} \int_{\mathbb{T}} P(z)z^{-4}e^{-\frac{2}{z^2}}dz.
\]
\[ J^S = x^3 J + \lambda (4x^2 - 4) \frac{d}{dx} - \lambda x^3 \frac{d^2}{dx^2}. \]

- \( H = \deg \tilde{\phi} = 3 \) and

\[ x^3 S_n(x) = Q^S_{n+3}(x) + \sum_{\nu=n-3}^{n+2} \mu^S_{n,\nu} Q^S_{\nu}(x), \]

\[ J^S Q^S_n(x) = S_{n+3}(x) + \sum_{\nu=n-3}^{n+2} \vartheta^S_{n,\nu} S_{\nu}(x), \]

\[ J^S Q^S_n(x) = Q^S_{n+3}(x) + \sum_{\nu=n-3}^{n+2} \omega^S_{n,\nu} Q^S_{\nu}(x). \]
Some references

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  Semiclassical quasi-orthogonal polynomials. A general calculus approach. In process.