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On the determinant of a sum of matrices

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Scheme of the talk

1. Overview
2. Main result
3. An application: The elementary symmetric functions

Basic overview

The determinant of a Matrix A can be determinate by using:

1. *The Laplace expansion*:

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{i,j} \det(A_{i,j}) = \sum_{i=1}^{n} (-1)^{i+j} a_{i,j} \det(A_{i,j}).$$

2. *The alternating sum*:

$$\det(A) = \sum_{\sigma \in P_n} \operatorname{sgn}(\sigma)\, a_{1,\sigma(1)} a_{2,\sigma(2)} \cdots a_{n,\sigma(n)},$$

The characteristic polynomial and the elementary symmetric functions:

1. Definition:

$$\chi_A(\lambda) \stackrel{\text{def}}{=} \det(\lambda I_n - A) = \prod_{j=1}^{n} (\lambda - \lambda_j).$$

2. The Elementary symmetric functions: For $k = 1, 2, \ldots, n$,

$$S_k \equiv S_k(\lambda_1, \ldots, \lambda_n) \stackrel{\text{def}}{=} \sum_{1 \le i_1 \le i_2 \le \cdots \le i_k} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k}$$

3. The Newton-Girard formulas: For any $m \in \mathbb{N}$,

$$m S_m = \sum_{j=1}^{m} (-1)^j S_j T_{m-j},$$

where $T_j(\lambda_1, \ldots, \lambda_n) \stackrel{\text{def}}{=} S_1(\lambda_1^j, \ldots, \lambda_n^j)$.

Main Result

Main Result:

Theorem: Given $A \in M_n$ and an integer N, with $N \geq n+1$. For any N-tuple $S = (A_1, A_2, \cdots, A_N)$, $A_i \in M_n$, $i = 1, \ldots, N$, the following relation holds:

$$\sum_{k=0}^{N} (-1)^k \sum_{\substack{\Omega \in \Sigma(S) \\ |\Omega| = k}} \det(A + \sum_{A_i \in \Omega} A_i) = 0,$$

understanding that $|\Omega| = k$ means that Ω is a formal sum with k summands, and that $A_i \in \Omega$ means that A_i is a summand in Ω.

$$\begin{aligned} \det(A_1 + A_2 + A_3 + A_4) = & \det(A_1 + A_2 + A_3) + \det(A_1 + A_2 + A_4) \\ & + \det(A_1 + A_3 + A_4) + \det(A_2 + A_3 + A_4) \\ & - \det(A_1 + A_2) - \det(A_1 + A_3) - \det(A_1 + A_4) \\ & - \det(A_2 + A_3) - \det(A_2 + A_4) - \det(A_3 + A_4) \\ & + \det(A_1) + \det(A_2) + \det(A_3) + \det(A_4). \end{aligned}$$

The elementary symmetric functions:

Corollary 1: Under the conditions of the Theorem. For any index sets $\alpha, \beta \subseteq \{1, 2, \dots, n\}$ of size τ, $N \geq \tau + 1$,

$$\sum_{k=0}^{N} (-1)^k \sum_{\substack{\Omega \in \Sigma(S) \\ |\Omega| = k}} \det(A(\alpha, \beta) + \sum_{A_i \in \Omega} A_i(\alpha, \beta)) = 0.$$

Definition: The k-th elementary symmetric function is defined as:

$$S_k(A) \stackrel{\text{def}}{=} \sum_{|\alpha| = k} \det(A(\alpha, \alpha)).$$

Corollary 2: For any non-negative τ, N, $N \geq \tau + 1$,

$$\sum_{k=0}^{N} (-1)^k \sum_{\substack{\Omega \in \Sigma(S) \\ |\Omega| = k}} S_\tau(A + \sum_{A_i \in \Omega} A_i) = 0.$$

A remark about the Theorem

The Theorem is optimal with respect to the range of N, i.e. for any positive integer n, it is possible to find n-tuples of M_n so that the equality given in Theorem is not true.
For instance, taking

$$A_i = \operatorname{diag}(e_i), \quad i = 1, 2, \ldots, n, \quad A = xe_1,\ x \in \mathbb{R},$$

where $\{e_1, e_2, \ldots, e_n\}$ is the canonical basis of $\mathbb{R}^n$, it is straightforward to check that

$$\sum_{k=0}^{n} (-1)^k \sum_{\substack{\Omega \in \Sigma(S) \\ |\Omega| = k}} \det(A + \sum_{A_i \in \Omega} A_i) = (-1)^n (1 + x - x) \neq 0.$$

An application: The elementary symmetric functions

Some computation:

$$S_2(A_1+A_2) = S_2(A_1)+S_2(A_2)+S_1(A_1)S_1(A_2)-S_1(A_1A_2),$$

$$\begin{aligned} S_3(A_1+A_2) = \; & S_3(A_1)+S_3(A_2)-S_1(A_1+A_2)S_1(A_1A_2) \\ & +S_1(A_1)S_2(A_2)+S_1(A_2)S_2(A_1) \\ & +S_1(A_1^2A_2)+S_1(A_1A_2^2). \end{aligned}$$

Question: Can we, in general, express S_k of a set of j n-by-n matrices, $j \leq k$, in an analogous way?.

Answer: YES.

More identities

$$S_k(A_1+A_2+\cdots+A_{k+l}) = \sum_{j=0}^{k}(-1)^j \binom{j+l-1}{l-1} \sum_{|\Omega|=k-j} S_k\left(\sum_{A_i\in\Omega} A_i\right).$$

$$S_1(A^m) = \det\begin{pmatrix} S_1(A) & 1 & 0 & \cdots & 0 \\ 2S_2(A) & S_1(A) & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ mS_m(A) & S_{m-1}(A) & S_{m-2}(A) & \cdots & S_1(A) \end{pmatrix}.$$

Two conjectures on elementary symmetric functions and positive definite matrices

The Bessis-Moussa-Villani conjecture: *The polynomial* $p(t) := Tr((A+tB)^m) \in \mathbb{R}[t]$*, has only nonnegative coefficients whenever* $A, B \in M_r$ *are positive semidefinite matrices.*

In fact, some numerical evidences and the Newton-Girard formulas suggested to us to consider a more general conjecture.

Positivity Conjecture :
The polynomial $S_k((A+tB)^l m) \in \mathbb{R}[t]$*, has only nonnegative coefficients whenever* $A, B \in M_r$ *are positive semidefinite matrices for every* $k = 0, 1, \ldots, r$.

Finally ....

THANKS FOR YOUR ATTENTION !!