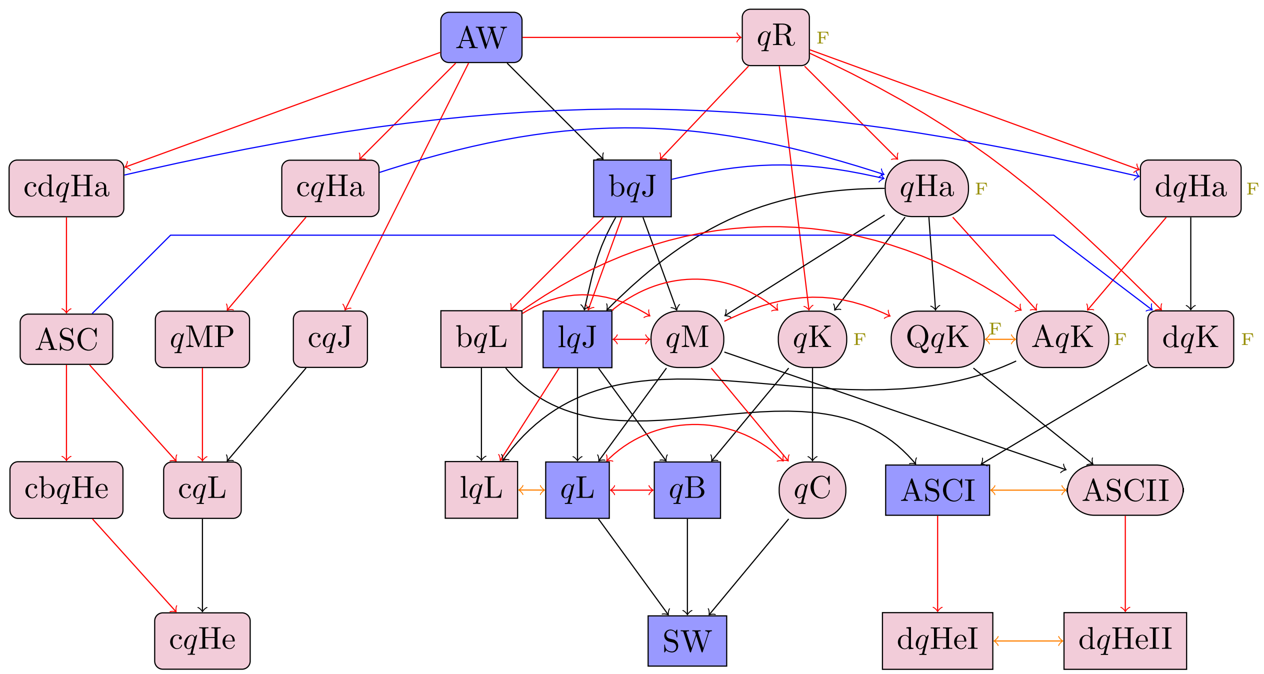
The Classical Basic Hypergeometric Orthogonal Polynomials



Conociendo mejor a los q-polinomios

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Universidad de Alcalá

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Extensions of discrete classical orthogonal polynomials beyond the orthogonality

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A B S T R A C T

It is well-known that the family of Hahn polynomials $\{h_n^{\alpha,\beta}(x;N)\}_{n\geq 0}$ is orthogonal with respect to a certain weight function up to degree N. In this paper we prove, by using the three-term recurrence relation which this family satisfies, that the Hahn polynomials can be characterized by a Δ-Sobolev orthogonality for every n and present a factorization for Hahn polynomials for a degree higher than N.

We also present analogous results for dual Hahn, Krawtchouk, and Racah polynomials and give the limit relations among them for all $n \in \mathbb{N}_0$. Furthermore, in order to get

q-Classical Orthogonal Polynomials: A General Difference Calculus Approach

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Abstract It is well known that the classical families of orthogonal polynomials are characterized as the polynomial eigenfunctions of a second order homogeneous linear differential/difference hypergeometric operator with polynomial coefficients.

In this paper we present a study of the classical orthogonal polynomials sequences, in short classical OPS, in a more general framework by using the differential (or difference) calculus and Operator Theory. The Hahn's Theorem and a characterization theorem for the q-polynomials which belongs to the q-Askey and Hahn tableaux are proved. Finally, we illustrate our results applying them to some known families of orthogonal q-polynomials.

Theorem 4.3 *Let (p_n) be an OPS with respect to $\rho(s)$ on the lattice $x(s)$ defined in (14) and let $\sigma(s)$ be such that (19) holds. Then the following statements are equivalent:*

1. *(p_n) is q-classical.*
2. *The sequence $(\Delta^{(1)} p_n)$ is an OPS with respect to the weight function $\rho_1(s) = \sigma(s+1)\rho(s+1)$ where ρ satisfies (15).*
3. *For every integer k, the sequence $(\mathcal{R}_n(\rho_k(s), x_k(s))(1))$ is an OPS with respect to the weight function $\rho_k(s)$ where $\rho_0(s) = \rho(s)$, $\rho_k(s) = \rho_{k-1}(s+1)\sigma(s+1)$, and ρ satisfies (15).*
4. *(Second order linear difference equation): (p_n) satisfies the following second order linear difference equation of hypergeometric type*

$$\sigma(s)\frac{\Delta}{\nabla x_1(s)}\frac{\nabla p_n(s)}{\nabla x(s)} + \tau(s)\frac{\Delta p_n(s)}{\Delta x(s)} + \lambda_n p_n(s) = 0, \tag{20}$$

where $\widehat{\sigma}(s) = \sigma(s) + \frac{1}{2}\tau(s)\nabla x_1(s)$ and $\tau(s)$ are polynomials on $x(s)$ of degree at most 2 and 1, respectively, and λ_n is a constant.

5. *(p_n) can be expressed in terms of the Rodrigues Operator as follows*

$$p_n(s) = B_n \mathcal{R}_n(\rho(s), x(s))(1) = \frac{B_n}{\rho(s)}\frac{\nabla}{\nabla x_1(s)}\frac{\nabla}{\nabla x_2(s)}\cdots\frac{\nabla}{\nabla x_n(s)}(\rho_n(s)), \tag{21}$$

where B_n is a non zero constant.

6. *(Second structure relation) There exist three sequences of complex numbers, (e_n), (f_n), and (g_n), such that the following relation holds for every $n \geq 0$, with the convention $p_{-1} = 0$,*

$$\mathcal{M} p_n(x(s)) = e_n \frac{\Delta p_{n+1}(s)}{\Delta x(s)} + f_n \frac{\Delta p_n(s)}{\Delta x(s)} + g_n \frac{\Delta p_{n-1}(s)}{\Delta x(s)},$$

where $\mathcal{M}$ is the forward arithmetic mean operator:

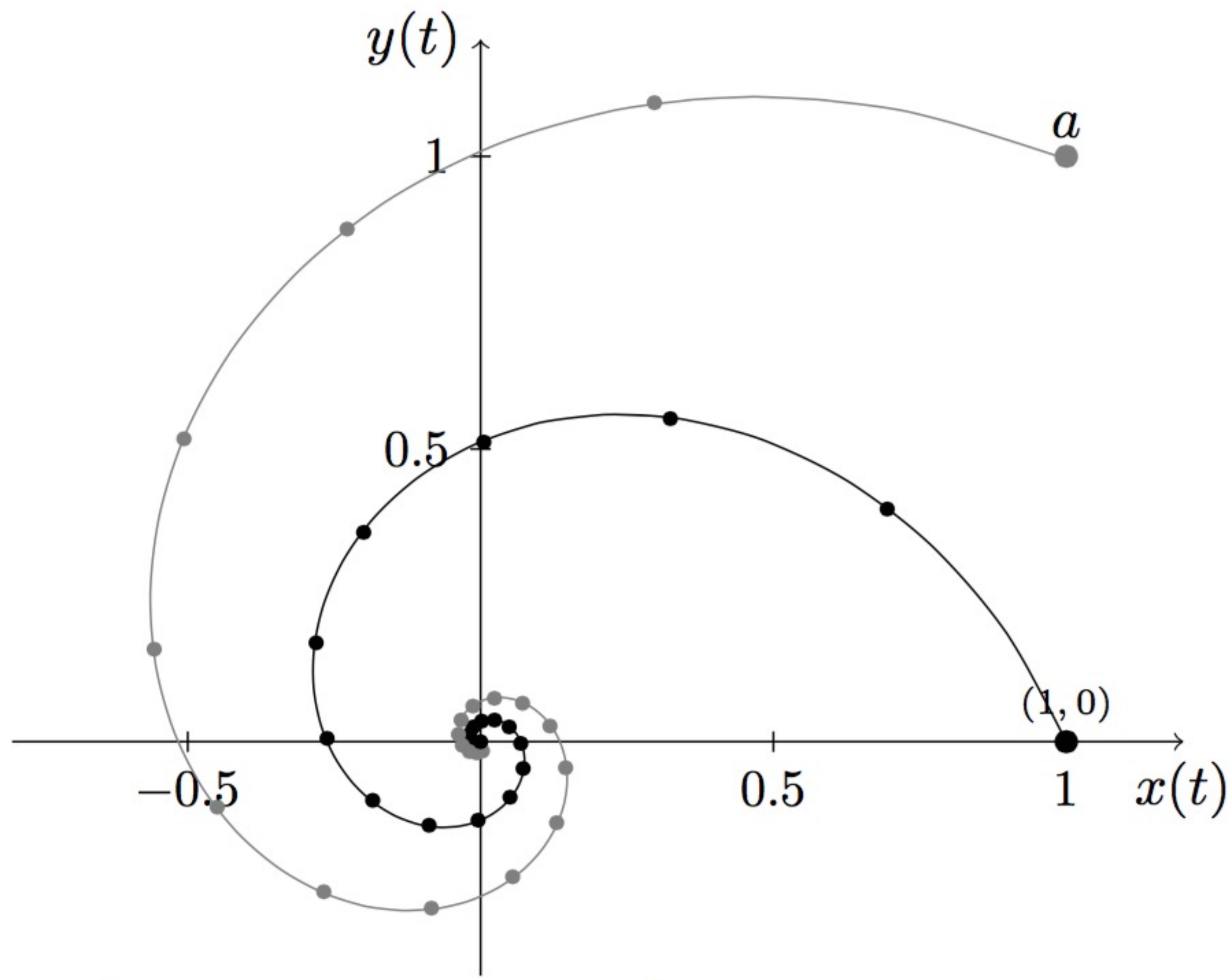
$$\mathcal{M} f(s) := \frac{f(s+1) + f(s)}{2},$$

$e_n \neq 0$, $g_n \neq \gamma_n$ for all $n \geq 0$, and γ_n is the corresponding coefficient of the three-term recurrence relation [25]

$$x(s) p_n(s) = \alpha_n p_{n+1}(s) + \beta_n p_n(s) + \gamma_n p_{n-1}(s), \quad n \geq 0.$$

Theorem 4.4 *Under the hypothesis of Theorem 4.3 the following statements are equivalent:*

(i) *(p_n) is q-classical.*
(ii) *$(\Delta^{(1)} p_{n+1})$ is a OPS.*



The lattice $\{q^k : k \in \mathbb{N}_0\} \cup \{(1+i)q^k : k \in \mathbb{N}_0\}$ with $q = 0.8\exp(\pi i/6)$.

The Al-Salam-Carlitz polynomials $U_n^{(a)}(x;q)$ were introduced by W. A. Al-Salam and L. Carlitz in [1] as follows

$$U_n^{(a)}(x;q) := (-a)^n q^{\binom{n}{2}} \sum_{k=0}^{n} \frac{(q^{-n};q)_k (x^{-1};q)_k}{(q;q)_k} \frac{q^k x^k}{a^k}. \tag{1.1}$$

In fact, these polynomials have a Rodrigues-type formula [4, (3.24.10)]

$$U_n^{(a)}(x;q) = \frac{a^n q^{\binom{n}{2}} (1-q)^n}{q^n \omega(x;a;q)} (\mathscr{D}_{q^{-1}})^n [\omega(x;a;q)], \qquad \omega(x;a;q) := (qx;q)_\infty (qx/a;q)_\infty, \tag{1.2}$$

where the q-Pochhammer symbol is defined as follows

$$(z;q)_0 := 1, \quad (z;q)_n := \prod_{k=0}^{n-1} (1-zq^k),$$

$$(z;q)_\infty := \prod_{k=0}^{\infty} (1-zq^k), \quad |z|<1.$$

and the q-derivative operator is defined by

$$(\mathscr{D}_q f)(z) := \begin{cases} \dfrac{f(qz)-f(z)}{(q-1)z} & \text{if } q \neq 1 \wedge z \neq 0, \\ f'(z) & \text{if } q = 1 \vee z = 0. \end{cases}$$

$$\int_a^1 U_n^{(a)}(x;q) U_m^{(a)}(x;q) (qx, qx/a;q)_\infty d_q x = (-a)^n (1-q)(q;q)_n (q;q)_\infty (a;q)_\infty (q/a;q)_\infty q^{\binom{n}{2}} \delta_{nm},$$

where the q-Jackson integral is defined as

$$\int_0^a f(x) d_q x := a(1-q) \sum_{n=0}^{\infty} f(aq^n), \quad \int_a^b f(x) d_q x = \int_0^b f(x) d_q x - \int_0^a f(x) d_q x.$$

Proof. Let $0 < |q| < 1$, and $a \in \mathbb{C}$, $a \neq 0, 1$. We are going to express the q-Jackson integral (2.1) as the difference of the two infinite sums and apply the identity

$$\sum_{k=0}^{M} f(q^k)(\mathscr{D}_{q^{-1}}g)(q^k)q^k = \frac{f(q^M)g(q^M) - f(q^{-1})g(q^{-1})}{q^{-1}-1} - \sum_{k=0}^{M} g(q^{k-1})(\mathscr{D}_{q^{-1}}f)(q^k)q^k. \tag{2.2}$$

Let $n \geq m$ then, for one side since $\omega(q^{-1}; a; q) = 0$ and using the identity [4, (14.24.9))], we get

$$\begin{aligned}
&\sum_{k=0}^{\infty} \omega(q^k; a; q) U_m^{(a)}(q^k; q) U_n^{(a)}(q^k; q) q^k \\
&= \lim_{M\to\infty} \frac{a(1-q)}{q^{2-n}} \sum_{k=0}^{M} \mathscr{D}_{q^{-1}}[\omega(q^k; a; q) U_{n-1}^{(a)}(q^k; q)] U_m^{(a)}(q^k; q) q^k \\
&= \lim_{M\to\infty} a q^{n-1} U_m^{(a)}(q^M; q) U_{n-1}^{(a)}(q^M; q) \omega(q^M; a; q) \\
&+ a q^{n-1}(q^m - 1) \sum_{k=0}^{M-1} \omega(q^k; a; q) U_{n-1}^{(a)}(q^k; q) U_{m-1}^{(a)}(q^k; q) q^k.
\end{aligned}$$

Following an analogous process than before and since $\omega(aq^{-1}; a; q) = 0$ we get

$$\begin{aligned}
&\sum_{k=0}^{\infty} \omega(aq^k; a; q) U_m^{(a)}(aq^k; q) U_n^{(a)}(aq^k; q) aq^k \\
&= \lim_{M\to\infty} \frac{U_m^{(a)}(aq^M; q) U_{n-1}^{(a)}(aq^M; q) \omega(aq^M; a; q)}{q^{-1}-1} \\
&+ \frac{a(q^m - 1)}{q^{1-n}} \sum_{k=0}^{M-1} \omega(aq^k; a; q) U_{n-1}^{(a)}(aq^k; q) U_{m-1}^{(a)}(aq^k; q) aq^k.
\end{aligned}$$

Therefore if $m < n$ then

$$\int_a^1 U_n^{(a)}(x; q) U_m^{(a)}(x; q) (qx, qx/a; q)_\infty d_q x = 0.$$

And for $n=m$ following the same idea we have

$$\begin{aligned}
&\int_a^1 U_n^{(a)}(x;q)U_n^{(a)}(x;q)\omega(x;a;q)d_qx\\
&=\frac{a(q^n-1)}{q^{1-n}}\sum_{k=0}^{\infty}\left(\omega(q^k;a;q)\left(U_{n-1}^{(a)}(q^k;q)\right)^2 q^k - a\omega(aq^k;a;q)\left(U_{n-1}^{(a)}(aq^k;q)\right)^2 q^k\right)\\
&=(-a)^n(q;q)_n q^{\binom{n}{2}}\sum_{k=0}^{\infty}\left(\omega(q^k;a;q)q^k - a\,\omega(aq^k;a;q)q^k\right)\\
&=(-a)^n(q;q)_n(q;q)_\infty q^{\binom{n}{2}}\sum_{k=0}^{\infty}\left((q^{k+1}/a;q)_\infty - a(aq^{k+1};q)_\infty\right)\frac{q^k}{(q;q)_k}\\
&=(-a)^n(q;q)_n(q;q)_\infty q^{\binom{n}{2}}(a;q)_\infty(q/a;q)_\infty.
\end{aligned}$$

Due the normality of this polynomial sequence, i.e., $\deg U_n^{(a)}(x;q)=n$ for all $n\in\mathbb{N}_0$, the uniqueness is straightforward hence the result holds. □

From this result and taking into account we the squared norm for the Al-Salam-Carlitz polynomials is known we got the following consequence which we could not find any reference about it.

Corollary 2.3. *Let $a,q\in\mathbb{C}\setminus\{0\}$, $|q|<1$. Then*

$$\sum_{k=0}^{\infty}\left((q^{k+1}/a;q)_\infty - a(aq^{k+1};q)_\infty\right)\frac{q^k}{(q;q)_k}=(a;q)_\infty(q/a;q)_\infty. \tag{2.3}$$

Theorem 2.4. *Let $a,q\in\mathbb{C}$, $a\neq 0,1$, $|q|>1$. Then the Al-Salam-Carlitz polynomials are the unique (up to a multiplicative constant) satisfying the property of orthogonality*

$$\begin{aligned}
&\int_\Gamma U_n^{(a)}(x;q^{-1})U_m^{(a)}(x;q^{-1})(q^{-1}x;q^{-1})_\infty(q^{-1}x/a;q^{-1})_\infty d_{q^{-1}}x=\\
&\qquad(-a)^n(1-q^{-1})(q^{-1};q^{-1})_n(q^{-1};q^{-1})_\infty q^{-\binom{n}{2}}(a;q^{-1})_\infty(q^{-1}/a;q^{-1})_\infty\,\delta_{m,n},
\end{aligned}\tag{2.4}$$

where Γ is the set of points $(aq^{-k})_{k=0}^{\infty}\cup(q^{-k})_{k=\infty}^{0}$.


obrigado
Dank U
Merci
mahalo
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спасибо
Grazie
Thank you
mauruuru
Takk
Gracias
Dziękuję
Děkuju
danke
Kiitos