q-Polynomials. Orthogonality in the complex plane and more

Roberto Costas Santos

Universidad de Alcalá

Joint work with H. Cohl, A. Soria-Lorente and F. Marcellan

Work supported by Ministerio de Economía y Competitividad of Spain, grant MTM2012-36732-C03-01

Copenhagen, DENMARK, August 26-29, 2015

The Real World is Complex - Congress in honor of Christian Berg

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Outline

1. The q-polynomials

* Classical Orthogonal Polynomials
* The support of the measure and the Jacobi Matrix
* q-polynomials. The relevant families

2. The examples

* The big q-Jacobi polynomials
* The Al-Salam-Carlitz polynomials

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Classical Orthogonal polynomials

- Let $\mathbf{u}$ be a linear functional.
- If $\mathbf{u}$ fulfills the distributional equation

$$\mathcal{D}(\phi\mathbf{u}) = \psi\mathbf{u}, \quad \deg\psi \leq 1,\ \deg\phi \leq 2$$

- Property of orthogonality: $\langle \mathbf{u}, P_n P_m \rangle = d_n^2 \delta_{n,m}$

  Three-term recurrence relation:

$$xP_n(x) = P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}(x)$$

- Integral representation with a weight function

$$\langle \mathbf{u}, P \rangle = \int_\Gamma P(z)\, d\mu(z), \qquad \Gamma \subset \mathbb{C}. \qquad d\mu(z) = \omega(z)\, dz$$

The weight function and the Favard's result

- The function $\omega(s)$ fulfills a Pearson-type difference eq.:

$$\phi(s+1)\omega(s+1) - \phi(s)\omega(s) = \big(x(s+1/2) - x(s-1/2)\big)\psi(s)$$

- The *q*-polynomials satisfy, in general, a property of orthogonality

$$\langle \mathbf{u}, P \rangle = \int_a^b P(z)\omega(z)\, d_q z$$

- Degenerate Favard's result: some gamma-coefficients of the TTRR are zero.

Orthogonality of *q*-polynomials for nonstandard parameters (with J. F. Sanchez-Lara) J. Approx. Theory 163 (2011), no. 9, 1246–1268.

The support of the measure and the Jacobi Matrix

Taking into account the TTRR

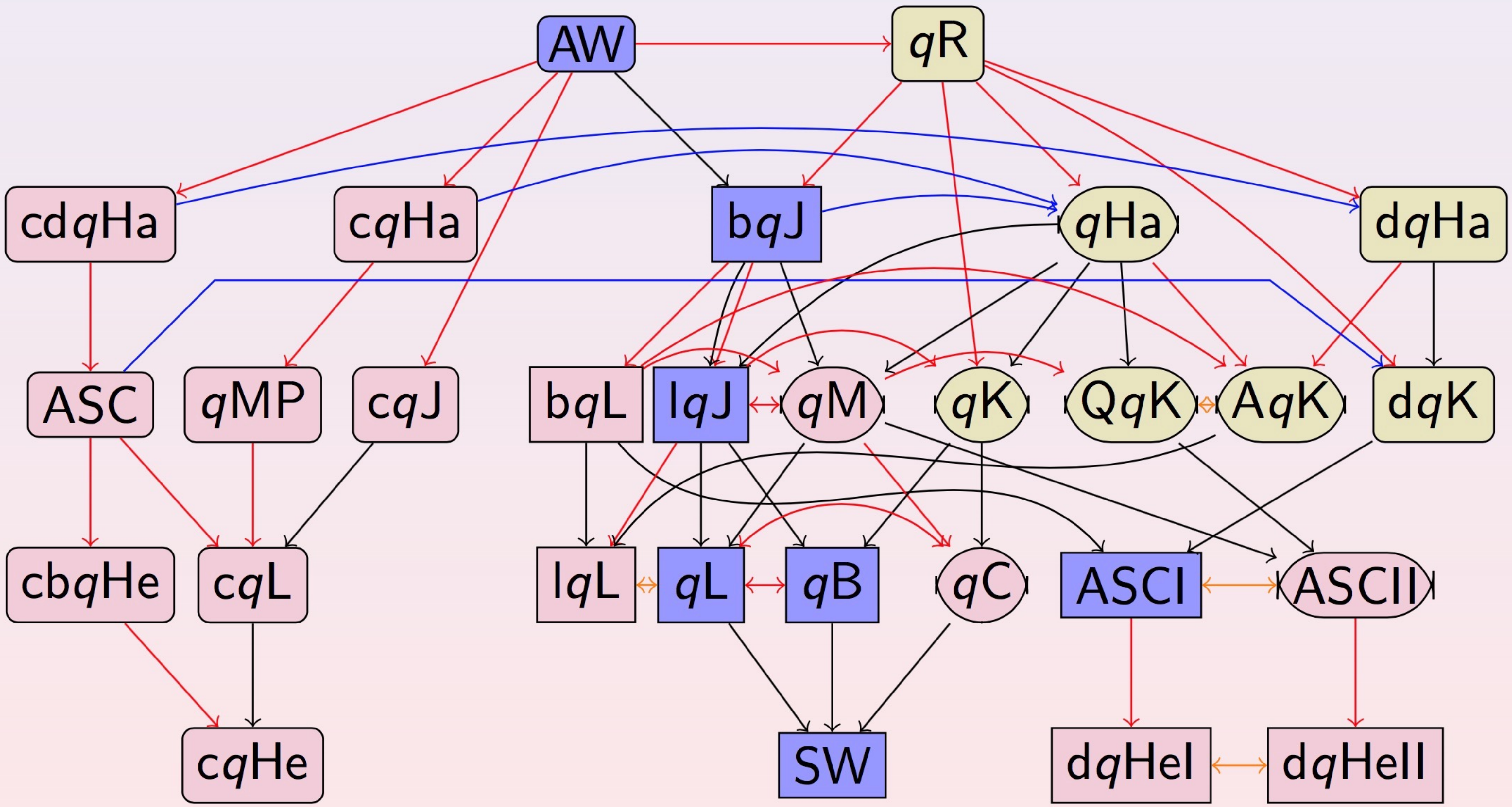
$$xP_n(x) = P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}(x)$$

one constructs the Jacobi matrix

$$J = \begin{pmatrix} \beta_0 & 1 & 0 & 0 & 0 & \cdots \\ \gamma_1 & \beta_1 & 1 & 0 & 0 & \cdots \\ 0 & \gamma_2 & \beta_2 & 1 & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

The spectrum of the N-by-N truncated Jacobi matrix are the zeros of $P_N(x)$ for all N.

Scheme of The Basic Hypergeometric OP



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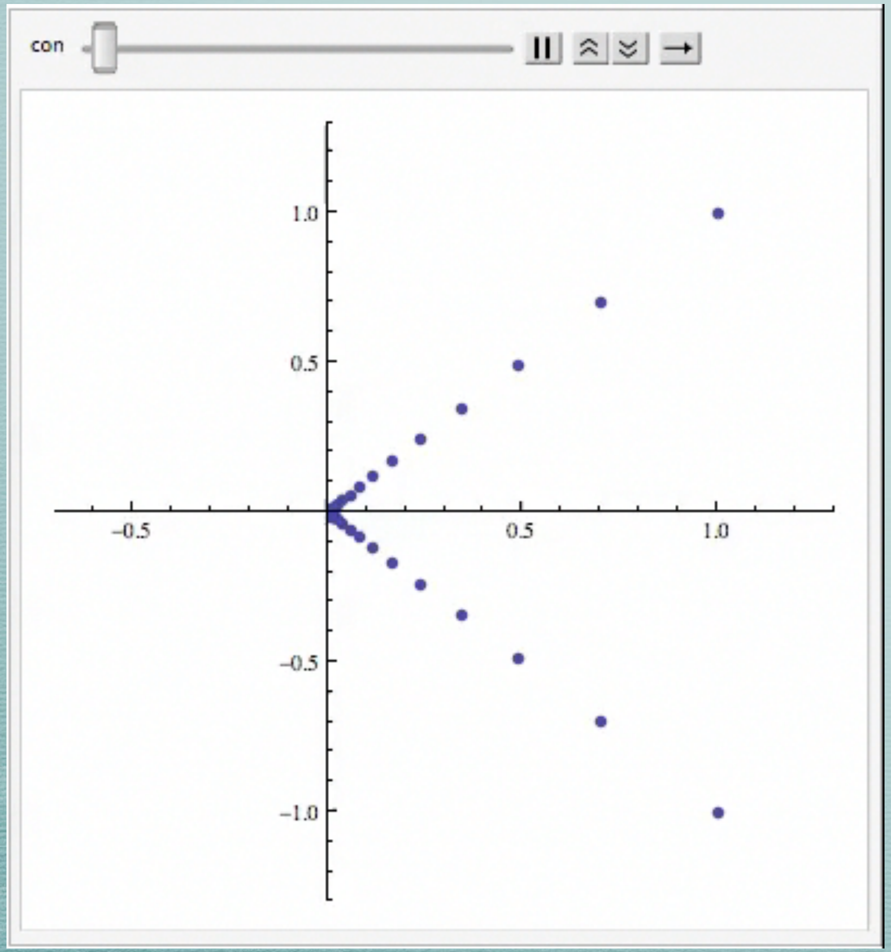
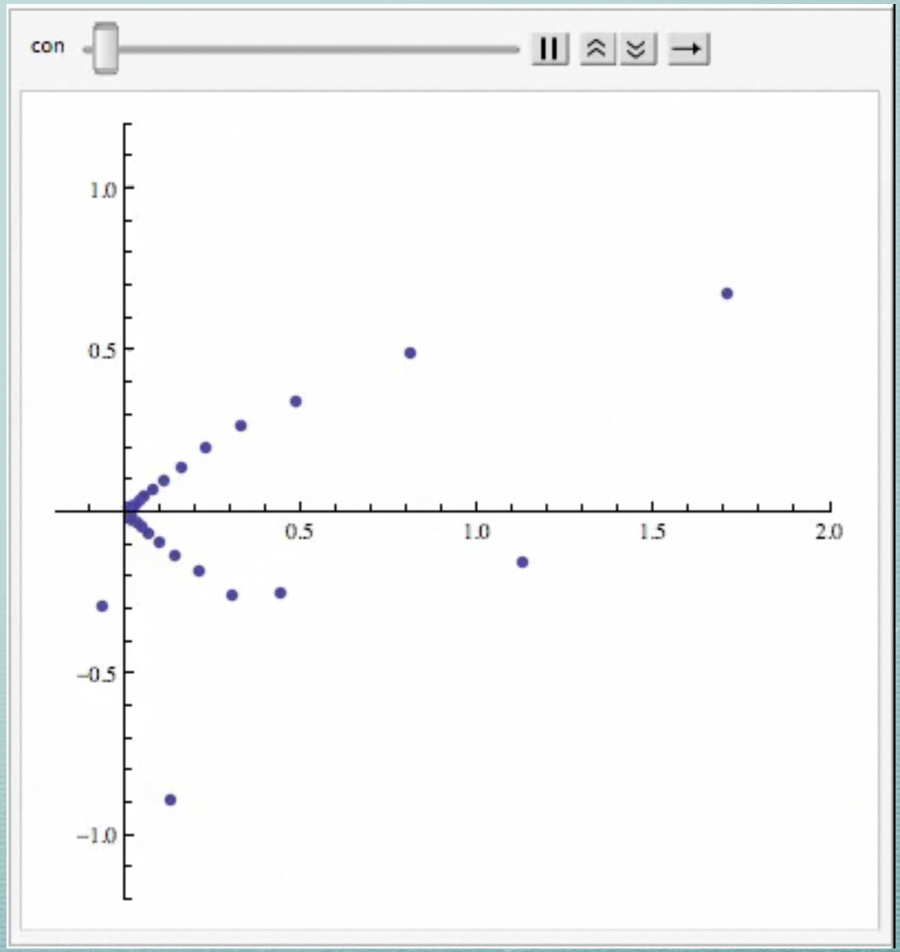
The big q-Jacobi polynomials

In this case $\phi(x) = (x - aq)(x - cq)/q$

For $a, b, c \in \mathbb{C},\ a \neq c,\ 0 < |q| < 1$



$aq = 1 + I,\ cq = 1 - I,\ q = 0.7$



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Krall. Mass point at 1+I

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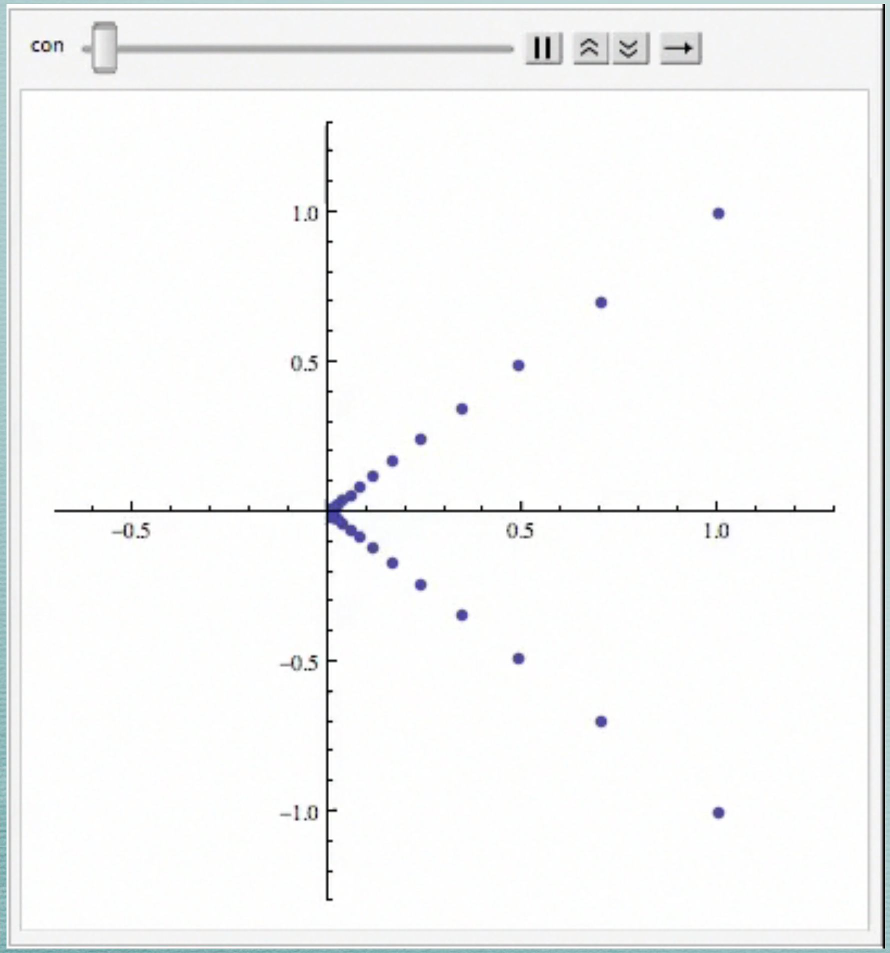

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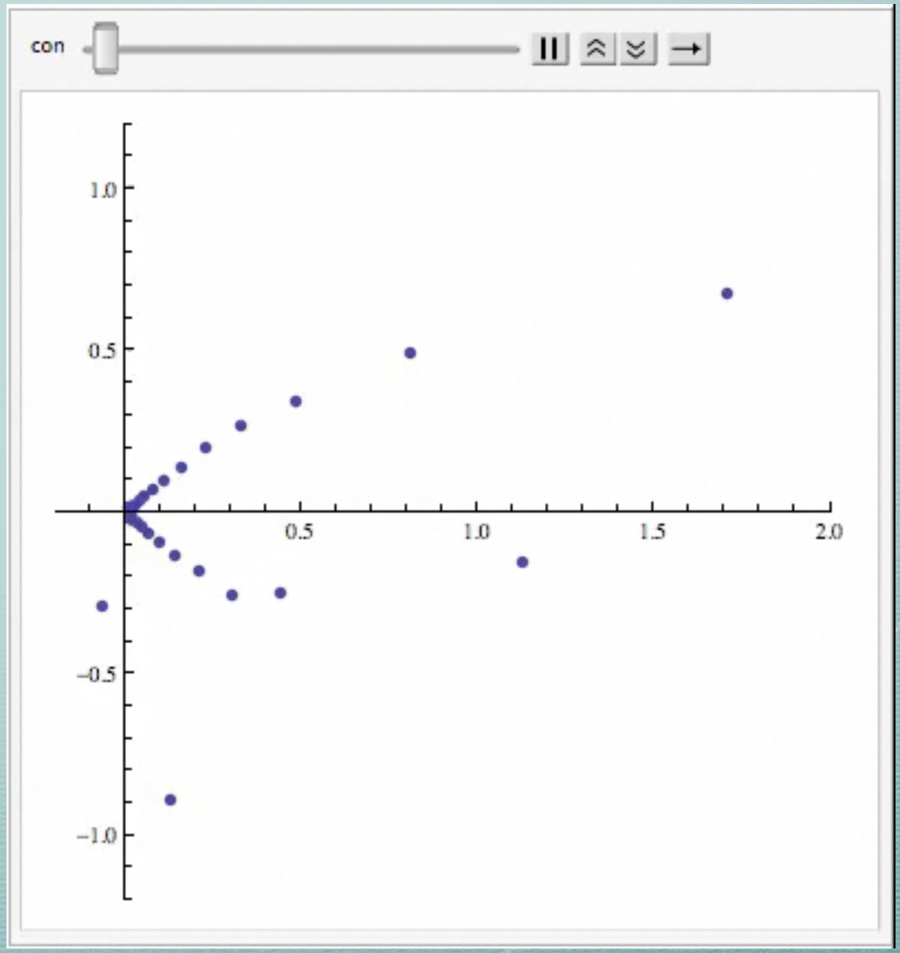

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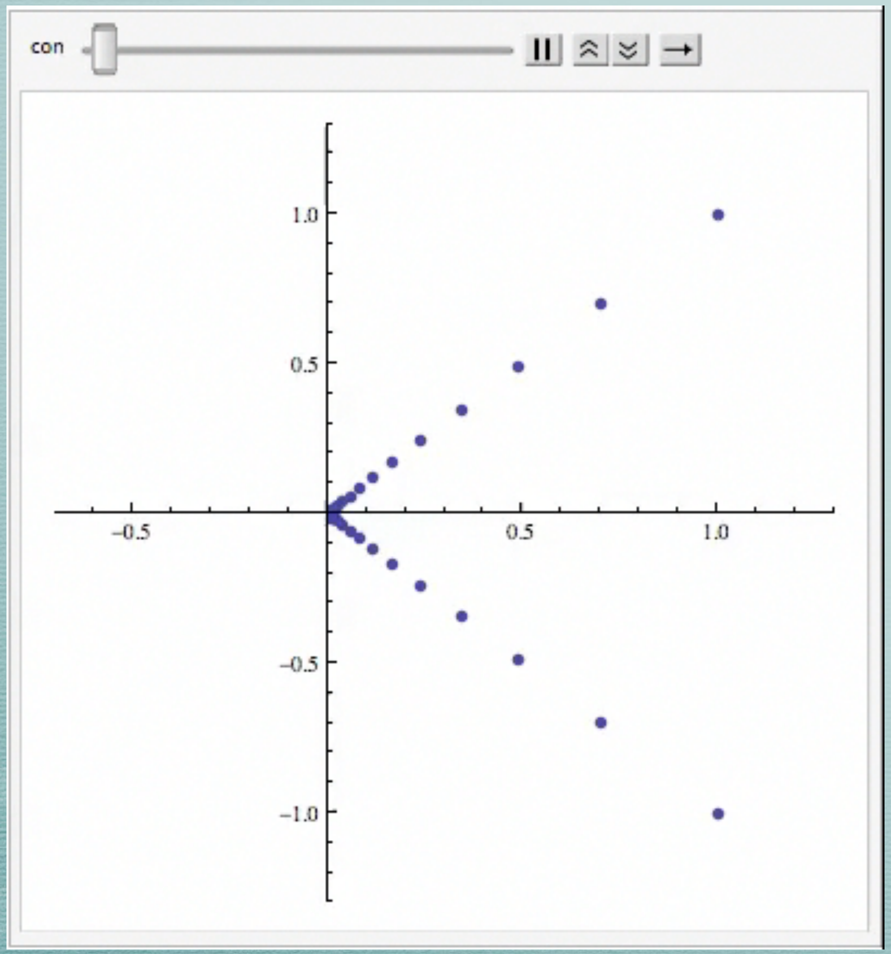

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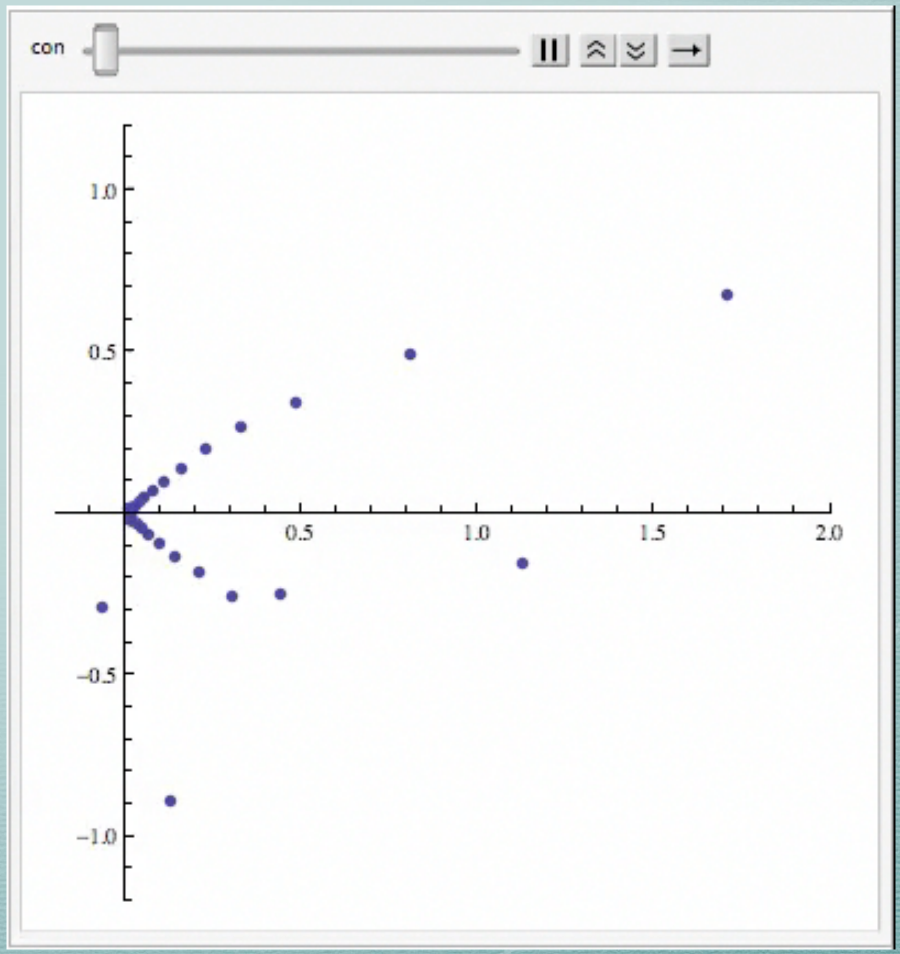

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The Al-Salam-Carlitz polynomials

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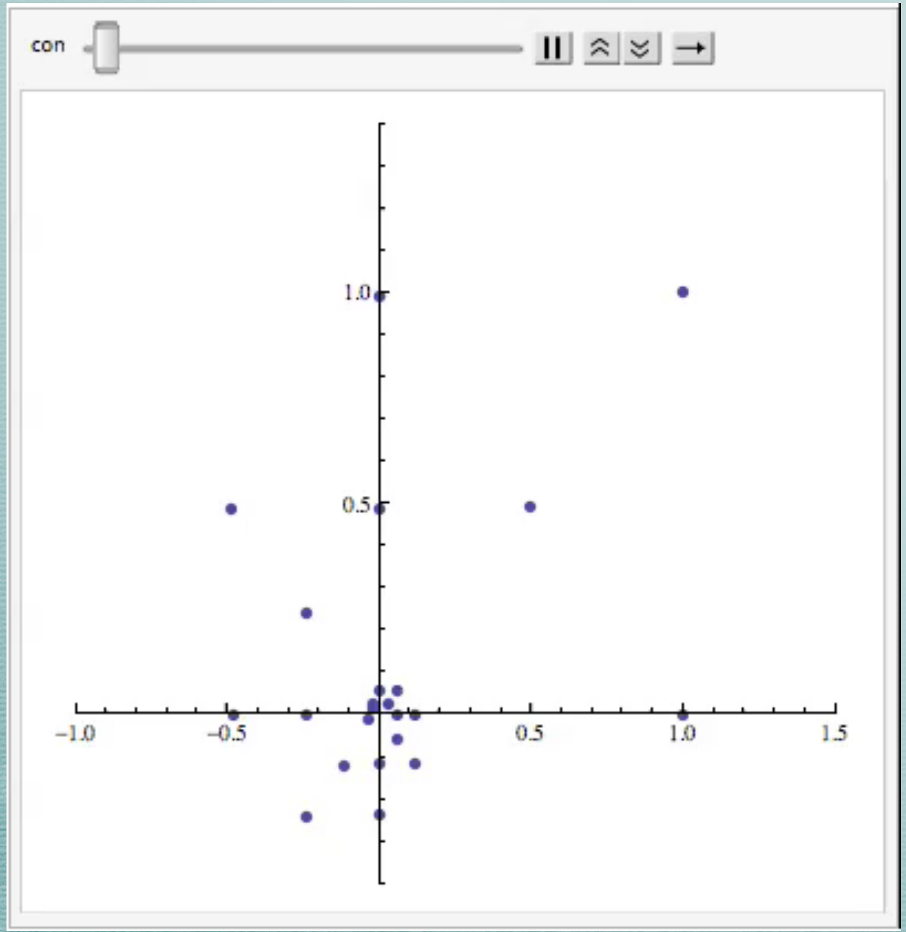

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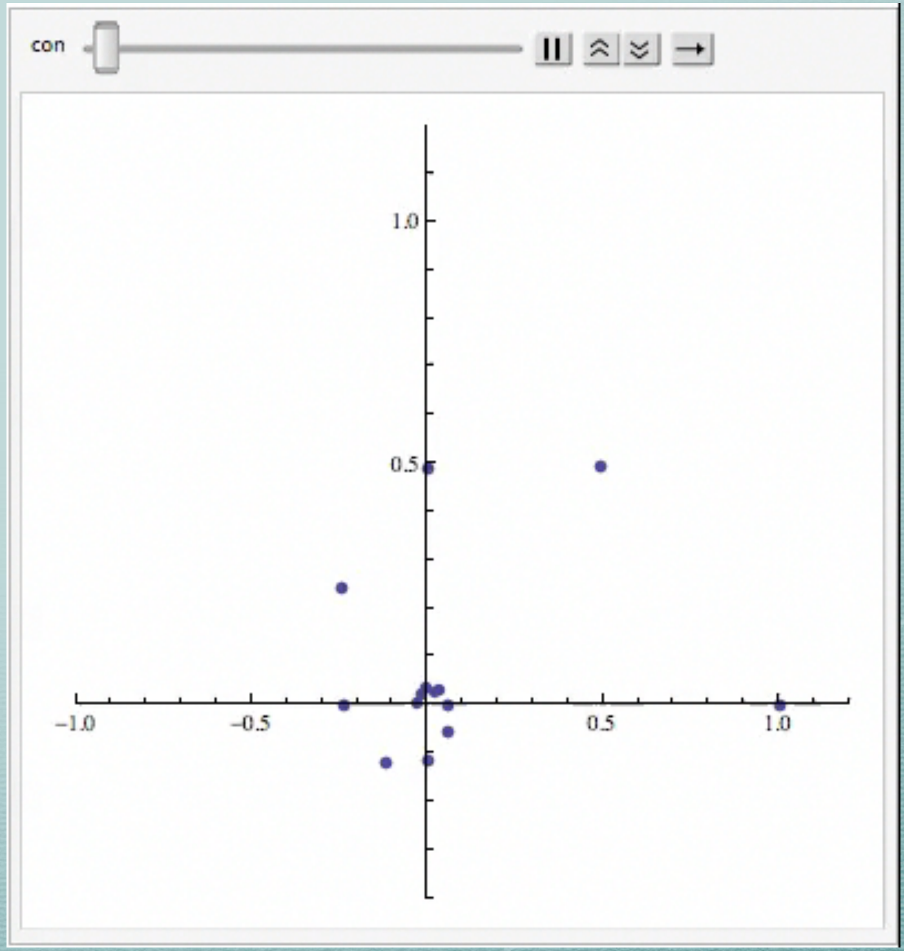

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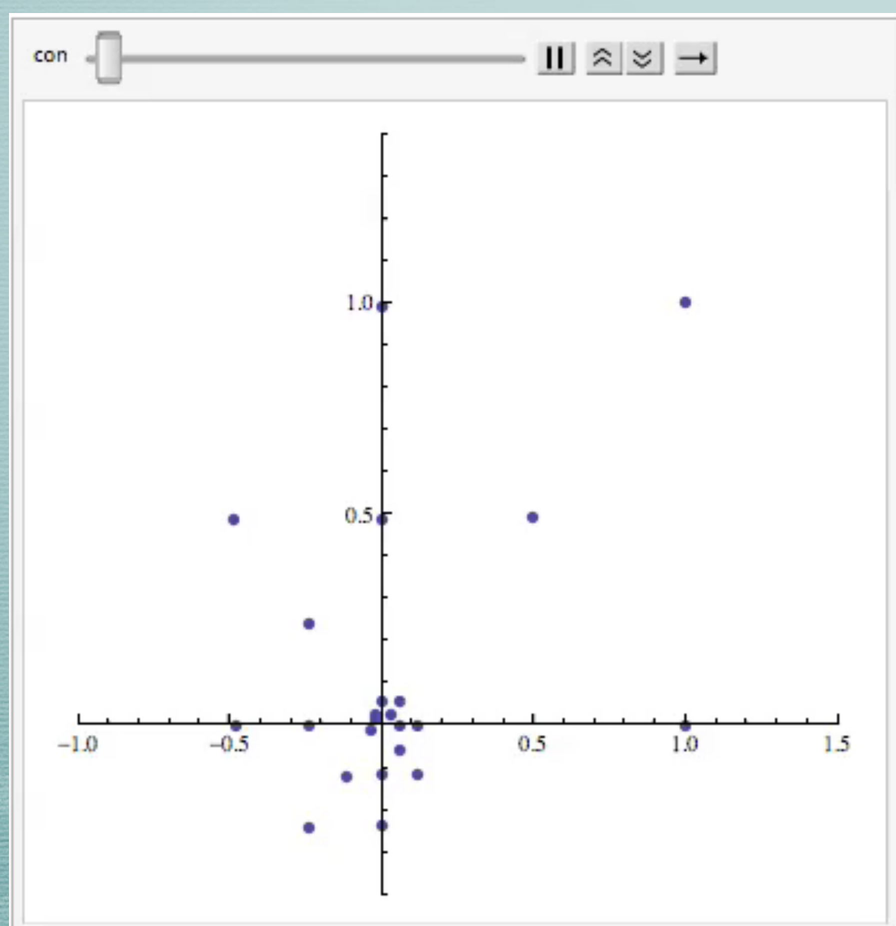

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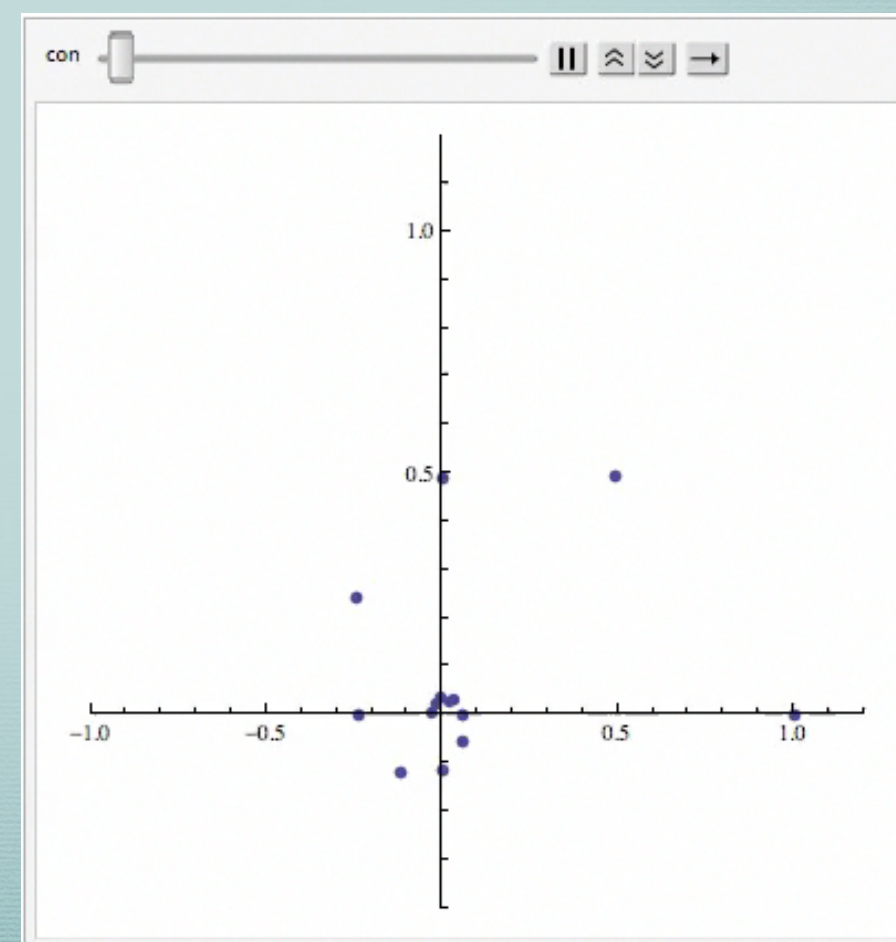

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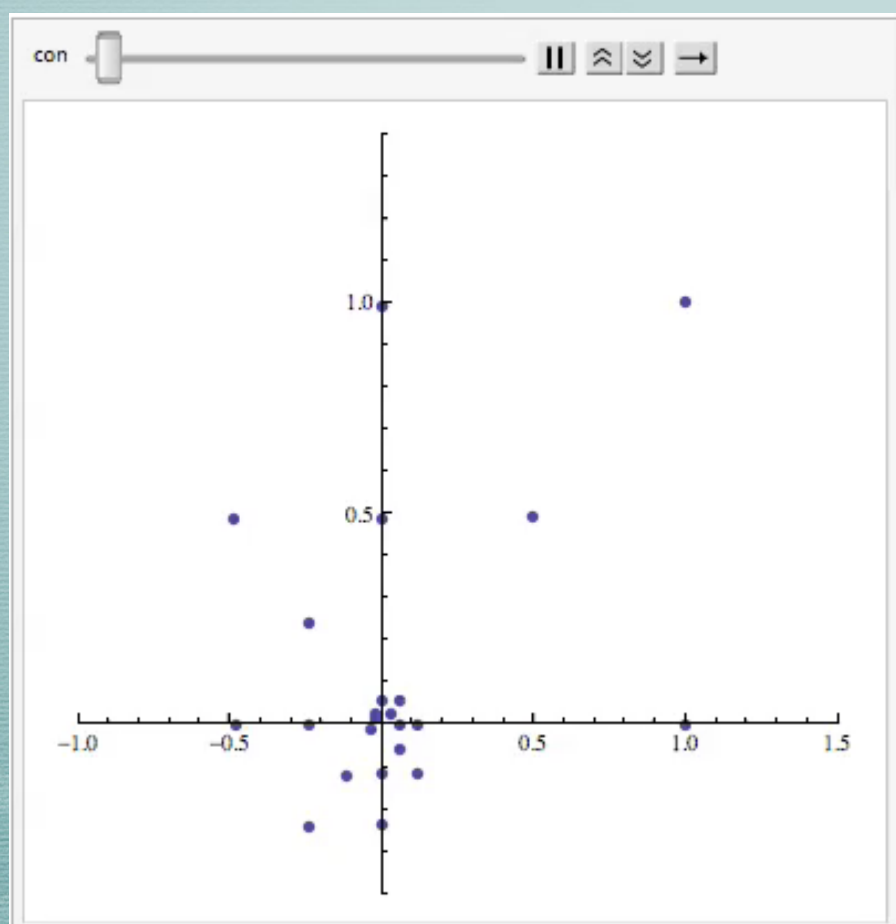

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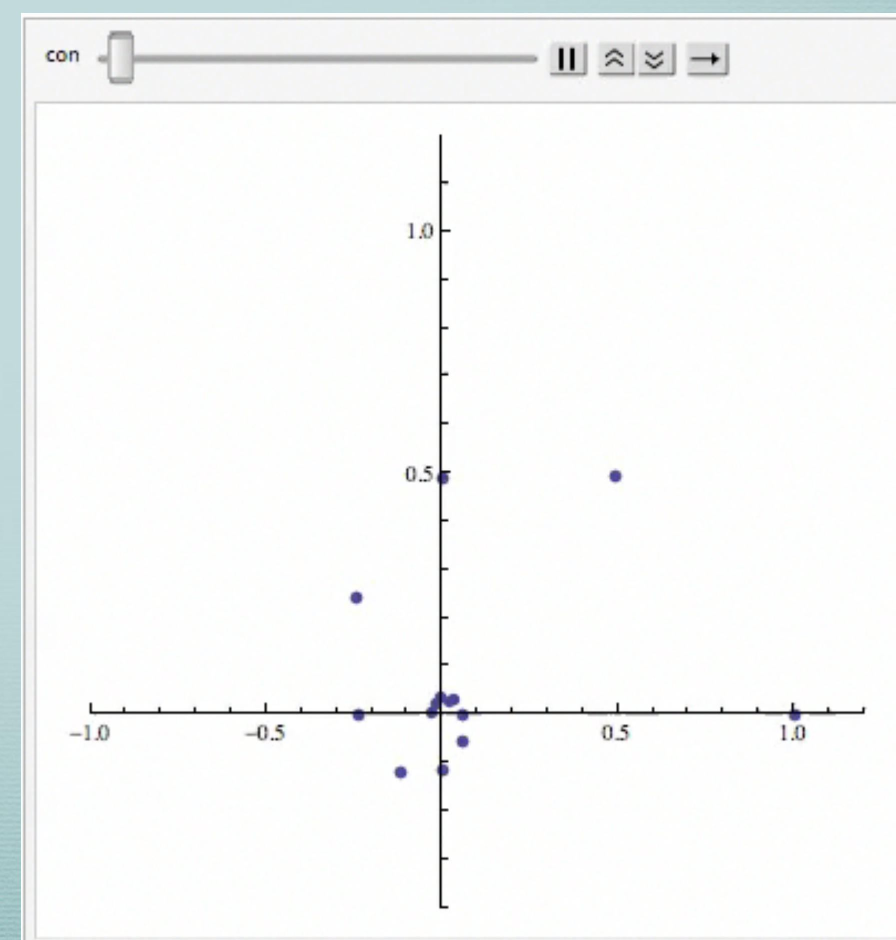

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Open problems

- Obtain ‘the minimal’ weight function for all the relevant families of q-polynomials.

- Obtain the behavior of the zeros of the Krall-type OP, as well the analytic properties when the mass we add is located in the complex plane.

- Give the orthogonality of the relevant families for the ‘bad cases’.

Thank you
for your attention