q-Polynomials for non-standard parameters.

Orthogonality and new identities

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Outline

- Basic definitions
- The scheme
- New identities (*structure relations*)
- Orthogonality for *q*-analogs of Laguerre polynomials

The basics

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- Let $\mathbf{u}$ be a (quasi-definite) linear functional

- If $\mathbf{u}$ fulfills the distributional equation

$$\mathcal{D}(\phi \mathbf{u}) = \psi \mathbf{u}, \quad \deg \psi \leq 1, \ \deg \phi \leq 2$$

$$P_n = \text{mops } \mathbf{u}$$

- Property of orthogonality: $\langle \mathbf{u}, P_n P_m \rangle = d_n^2 \delta_{n,m}$

- Three-term recurrence relation:

$$x P_n(x) = P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}(x)$$

The basics

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- Degenerate Favard's result, i.e. there exists some **N** such that $\gamma_{\mathbf{N}} = 0$.

Orthogonality of *q*-polynomials for nonstandard parameters (with J. F. Sanchez-Lara) J. Approx. Theory **163** (2011), 1246–1268.

- Integral representation with a weight function

$$\langle \mathbf{u}, P \rangle = \int_\Gamma P(z)\, d\mu(z), \qquad \Gamma \subset \mathbb{C}. \qquad d\mu(z) = \omega(z)\, dz$$

The basic hypergeometric OP scheme

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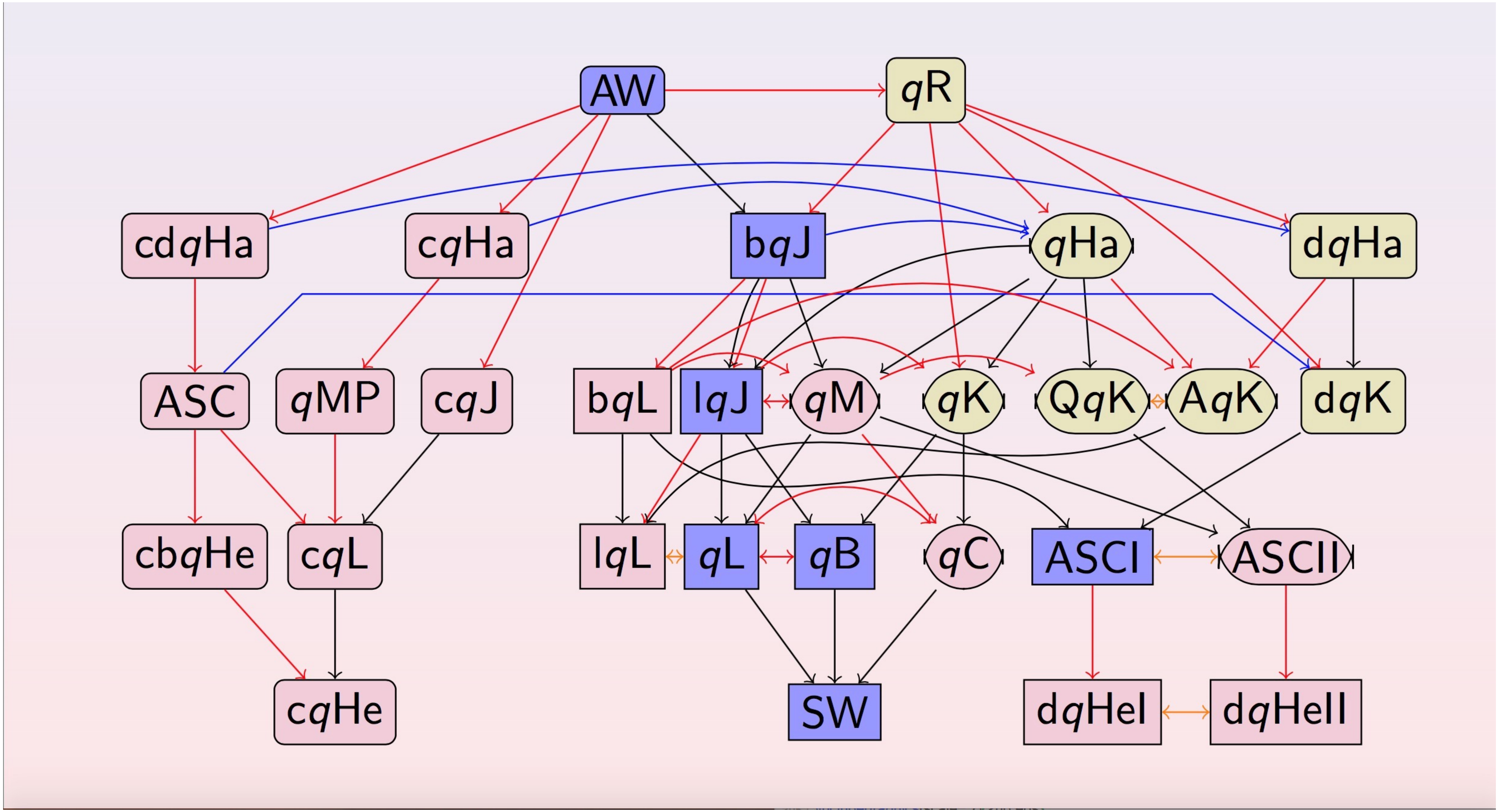

The orthogonality for the q-polynomials OPSFA14

- The q-polynomials satisfy, in general, a property of orthogonality

$$\langle \mathbf{u}, P \rangle = \int_a^b P(z)\omega(z)\, d_q z$$

where

$$\int_a^b f(x)\, d_q x := b(1-q)\sum_{n=0}^{\infty} f(bq^n)q^n - a(1-q)\sum_{n=0}^{\infty} f(aq^n)q^n$$

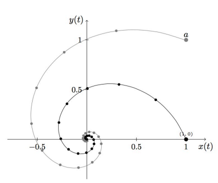

Fig. 2. The lattice $\{q^k : k \in \mathbb{N}_0\} \cup \{(1+i)q^k : k \in \mathbb{N}_0\}$ with $q = 4/5\exp(\pi i/6)$.

The new identities

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- Two different structure relations for COP

$$\phi(x)D(p_n(x)) = \tilde{a}_n p_{n+1}(x) + \tilde{b}_n p_n(x) + \tilde{c}_n p_{n-1}(x)$$

$$p_n(x) = \widehat{a}_n D(p_{n+1}(x)) + \widehat{b}_n D(p_n(x)) + \widehat{c}_n D(p_{n+1}(x))$$

But, in fact, ... they also fulfill

$$\phi^*(x)D^*(p_n(x)) = \tilde{a}^*_n p_{n+1}(x) + \tilde{b}^*_n p_n(x) + \tilde{c}^*_n p_{n-1}(x)$$

The new identities are connected with these two structure relations ….

The new identities

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- The classical cases

$$P_{n-1}^{(\alpha+1,\beta+1)}(x) \propto \frac{K_n^{\alpha,\beta}(x,-1)K_{n-1}^{\alpha,\beta}(1,-1) - K_n^{\alpha,\beta}(1,-1)K_{n-1}^{\alpha,\beta}(x,-1)}{x-1}$$

$$K_n^{\alpha,\beta}(x,-1) \propto \frac{P_n^{(\alpha,\beta)}(x)P_{n-1}^{(\alpha,\beta)}(-1) - P_n^{(\alpha,\beta)}(-1)P_{n-1}^{(\alpha,\beta)}(x)}{x+1}$$

$$L_{n-1}^{(\alpha+1)}(x) \propto \frac{L_n^{(\alpha)}(x)L_{n-1}^{(\alpha)}(0) - L_n^{(\alpha)}(0)L_{n-1}^{(\alpha)}(x)}{x}$$

The new identities

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- Some q-polynomial families

$$D^*(P_n(x;a,b,c;q)) \propto \frac{K_n(x;cq)K_{n-1}(aq;cq)-K_n(aq;cq)K_{n-1}(x;cq)}{x-aq}$$

$$K_n(x;cq) \propto \frac{P_n(x;a,b,c;q)P_{n-1}(cq;a,b,c;q)-P_n(cq;a,b,c;q)P_{n-1}(x;a,b,c;q)}{x-cq}$$

$$D(P_n(x;a,b,c;q)) \propto \frac{K_n(x,1)K_{n-1}(c/b,1)-K_n(c/b,1)K_{n-1}(x,1)}{bx-c}$$

$$K_n(x,1) \propto \frac{P_n(x;a,b,c;q)P_{n-1}(1;a,b,c;q)-P_n(1;a,b,c;q)P_{n-1}(x;a,b,c;q)}{x-1}$$

The new identities

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- Little q-Laguerre/Wall polynomials

$$D^*(p_n(x;a|q)) \propto \frac{K_n(x,1)K_{n-1}(0,1)-K_n(0,1)K_{n-1}(x,1)}{x}$$

$$K_n(x,1) \propto \frac{p_n(x;a|q)p_{n-1}(1;a|q)-p_n(1;a|q)p_{n-1}(x;a|q)}{x-1}$$

$$D(p_n(x;a|q)) \propto \frac{p_n(x;a|q)p_{n-1}(1-aq;a|q)-p_n(1-aq;a|q)p_{n-1}(x;a|q)}{x-1+aq}$$

And, what can we say about the another structure relation?

Orthogonality of *q*-polynomials for nonstandard parameters (with J. F. Sanchez-Lara) J. Approx. Theory **163** (2011), 1246–1268.

On Sobolev orthogonality for the generalized Laguerre polynomials. (M.T. Pérez, M. Piñar) *J. Approx. Theory* **86** (1996), 278--285.

Stating the problem(s)

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- **Problem 1:** given non-standard (complex) parameter a. Obtain an inner product *B1* with respect to the little *q*-Laguerre are orthogonal.

- **Problem 2:** given non-standard (complex) parameter a. Obtain an inner product *B2* with respect to the *q*-Laguerre are orthogonal.

- **Problem 3:** given non-standard (complex) parameters a, b. Obtain an inner product *B3* with respect to the big *q*-Laguerre are orthogonal.

The Laguerre case

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- In order to solve the **problem 1**, first we see what Mayte and Miguel did for the Laguerre case.

THEOREM 4.1. ...guerre poly-
nomials $\{L_n^{(\alpha)}(x)\}$ ...onal Sobolev
inner product $(\cdot, \cdot$

LEMMA 2.1. *Given $\alpha \in \mathbb{R}$, let $k \geqslant 0$, $n \geqslant 1$ be integers. Then*

(i) (*Recurrence relation*)

$$L_{-1}^{(\alpha)}(x) = 0, \qquad L_0^{(\alpha)}(x) = 1,$$

$$xL_n^{(\alpha)}(x) = L_{n+1}^{(\alpha)} + \beta_n^{(\alpha)} L_n^{(\alpha)}(x) + \gamma_n^{(\alpha)} L_{n-1}^{(\alpha)}(x),$$

where $\beta_n^{(\alpha)} = 2n + \alpha + 1$, $\gamma_n^{(\alpha)} = n(n + \alpha)$,

(ii) $$(L_n^{(\alpha)})'(x) = nL_{n-1}^{(\alpha+1)}(x),$$

(iii) $$L_n^{(\alpha)}(x) = L_n^{(\alpha+1)}(x) + nL_{n-1}^{(\alpha+1)}(x),$$

(iv) $$L_n^{(\alpha)}(x) = \sum_{i=0}^{k} (-1)^i \binom{k}{i} (L_n^{(\alpha-k)})^{(i)}(x).$$

More new identities

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- Little q-Laguerre polynomials

$$q^{1-n}\hat{P}_n(x;a|q) = (q-1)D^*(\hat{P}_n(x;a/q|q)) + q\hat{P}_n(x;a/q|q)$$

$$\frac{q^{-n}-a}{1-a}P_n(x;a|q) = P_n(x;a/q|q) + \frac{q-1}{q}D^*(P_n(x;a/q|q))$$

$$q^{-kn+k}\hat{P}_n(x;a|q) = \sum_{j=0}^{k}(q-1)^j q^{k-j}\begin{bmatrix} k \\ j \end{bmatrix}_q (D^*)^j(\hat{P}_n(xq^{-k+j};a/q^{-k}|q))$$

$$(f,g)_S^{k,a} := \langle \mathbf{u}_a^{lqL}, F(x)M(k)G(x)^T \rangle$$

More new identities

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The identities for the Big *q*-Laguerre polynomials are in the oven right now.

The will be ready soon!

THANK YOU
for your attention