Hypergeometric functions, difference equations and Wronskians

16th Orthogonal Polynomials, Special Functions and Applications

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Goal of this talk

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2. The main objective is to describe three fundamental tools that need to be handled to get rational approximations related to certain real numbers.

MOTIVATION:

Rational approximation of real numbers

Given a real number x, and two sequences of integers (a_n), (b_n) such that $b_n \neq 0$. a_n/b_n is a rational approximation of x if

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$$\left|x - \frac{a}{b}\right| < \frac{1}{b^2}.$$

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$$\left| x - \frac{a}{b} \right| < \frac{1}{b^2}.$$

- Apery in 1979 obtained recurrence relations which the numerators and denominators rational approximations of $\zeta(2)$ and $\zeta(3)$ satisfy respectively. [1]

- Later, Beukers in 1979 [2] and applying approximation Theory in a short note gave a basic explanation about such recurrence relations presented by Apery.

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- In such a note it can be verified that there are two tools that are fundamental:

  * the differential equations, and

  * the hypergeometric functions.

First tool:

The hypergeometric functions

Given two sequences $\{a_1, ..., a_r\}$ and $\{b_1, ..., b_s\}$ the hypergeometric series associated to these values is

$$ {}_rF_s\left(\begin{matrix} a_1, ..., a_r \\ b_1, ..., b_s \end{matrix} ; z \right) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_r)_k}{(b_1)_k \cdots (b_s)_k} \frac{z^k}{k!}, $$

where $b_1, ..., b_s$ is not a negative integer, and depending on the the values of r and s the range of converge changes, and the shifted factorial $(a)_k$, or Pochhammer symbol, is defined as

$$ (a)_k := a(a+1)\cdots(a+k-1). $$

First example: The Gauss function

The Gauss function

The sequence $f_n(z) = {}_2F_1(-n, b; c; z)$, with $c \neq 0$, fulfills the difference equation:

$$(c+n)f_{n+1}(z) = (c+2n-bz-nz)f_n(z) + n(z-1)f_{n-1}(z),$$

with initial conditions $f_0(z) = 1$ and $f_1(z) = 1 - bz/c$.

The Zeilberger's algorthm let us to obtain the difference equation certain hypergeometric series satisfies [5].

Second tool:

Difference equations

- The homogeneous are of the form:

$$p_0(n)u_n + p_1(n)u_{n+1}(n) + \cdots + p_d(n)u_{n+d} = 0$$

where $\deg p_k$ is a polynomial $k = 0, 1, ..., d$.

DIFFERENCE EQUATIONS [3]

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- In the cases we are going to consider (the one we can find in the literature related with such rational approximations) we have

$$p_d(n) = P(n+1)D_n, \qquad p_0(n) = P(n)N_n$$

where $\deg p_k(n)$ is constant, i.e. $\deg p_k(n) = T$, $k = 0, 1, \ldots, d$.

Second example: Rational approximation of $\zeta(3)$

- Let us consider the sequence

$$R_n = \sum_{k=n}^{\infty} (2k+n+2) \frac{(-k)_n (k+n+2)_n}{((k+1)_{n+1})^4}$$

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- Using the Zeilberger algorithm we get that R_n satisfies

$$n\, u_{n-1} + (2n+1)\left(17n^2+17n+5\right) u_n + (n+1)^5 u_{n+1} = 0,$$

with certain initial conditions.

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  with certain initial conditions.
- In fact, a_n and b_n satisfy the same difference equation.

- If we consider $S_n = (-1)^n (n!)^2 R_n$, then S_n fulfills

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- This one is the second order difference equation:

$$p_0(n-1)\, u_{n-1} + p_1(n) u_n + p_0(n) u_{n+1} = 0$$

$$p_0(n)\Delta\nabla u_n - \nabla p_0(n)\nabla u_n + \left(p_0(n-1) + p_1(n) + p_0(n)\right) u_n = 0$$

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$$\lambda_1 \approx -0.03 \qquad \lambda_2 \approx -33.97$$

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- Moreover

$$b_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2, \quad a_n = ??$$

In such example we have

$$R_n = a_n - b_n\zeta(3).$$

By the Poincare's theorem [4] we have

$$R_n = (-1)^n \frac{S_n}{(n!)^2} \approx \frac{(-\lambda_1)^n}{(n!)^2} \to 0,$$

$$a_n, b_n \approx \frac{(-\lambda_2)^n}{(n!)^2}$$

but the numbers a_n y b_n are not integers, but rational numbers.

$$\left|\frac{R_n}{b_n}\right| = \left|\zeta(3) - \frac{a_n}{b_n}\right|.$$

Numerical experiment

n	a_n	b_n	$\lvert R_n \rvert$
1	12	10	0.0205690316
2	$\frac{351}{8}$	$\frac{73}{2}$	0.00007696532519
3	$\frac{62531}{648}$	$\frac{1445}{18}$	$-1.579661956 * 10^{-7}$
4	$\frac{11424695}{82944}$	$\frac{33001}{288}$	$2.036802131 * 10^{-10}$
5	$\frac{35441662103}{259200000}$	$\frac{163801}{1440}$	$-1.799293875 * 10^{-13}$
6	$\frac{20637706271}{207360000}$	$\frac{858433}{10368}$	$1.156012181 * 10^{-16}$
7	$\frac{963652602684713}{17425497600000}$	$\frac{116861473}{2540160}$	$-5.642055399 * 10^{-20}$
8	$\frac{43190915887542721}{1784370954240000}$	$\frac{654716497}{32514048}$	$2.162249452 * 10^{-23}$

Third tool:

Discrete Wronskian

Firs relations with Wronskians

- Let us consider the expression:

$$R_n = a_0(n) + a_1(n)\eta_1 + a_2(n)\eta_2 + a_3(n)\eta_3,$$

where η_1, η_2 and η_3 are real numbers.

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where η_1, η_2 and η_3 are real numbers.

- Let us introduce the discrete Wronskian in such a case as

$$W_n(i,j,k) = \begin{vmatrix} a_i(n) & a_j(n) & a_k(n) \\ a_i(n+1) & a_j(n+1) & a_k(n+1) \\ a_i(n+2) & a_j(n+2) & a_k(n+2) \end{vmatrix},$$

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where $i \neq j \neq k$.

- Then

$$\begin{vmatrix} R_n & a_1(n) & a_2(n) \\ R_{n+1} & a_1(n+1) & a_2(n+1) \\ R_{n+2} & a_1(n+2) & a_2(n+2) \end{vmatrix} = W_n(0,1,2) + W_n(3,1,2)\eta_3$$

$$= W_n(0,1,2) + W_n(1,2,3)\eta_3.$$

In fact, we are going to define as well

$$W_n(R,1,2) = \begin{vmatrix} R_n & a_1(n) & a_2(n) \\ R_{n+1} & a_1(n+1) & a_2(n+1) \\ R_{n+2} & a_1(n+2) & a_2(n+2) \end{vmatrix}.$$

- Let us assume [3 equations, 4 polynomials]

$$\begin{aligned} p_0(n)a_1(n) + p_1(n)a_1(n+1) + p_2(n)a_1(n+2) + p_3(n)a_1(n+3) &= 0, \\ p_0(n)a_2(n) + p_1(n)a_2(n+1) + p_2(n)a_2(n+2) + p_3(n)a_2(n+3) &= 0, \\ p_0(n)a_3(n) + p_1(n)a_3(n+1) + p_2(n)a_3(n+2) + p_3(n)a_3(n+3) &= 0. \end{aligned}$$

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- If we work with such expressions in order to eliminate $p_1(n)$ we obtain

$$\begin{aligned} p_0(n)W_n(1,2) - p_2(n)W_{n+1}(1,2) + p_3(n)\begin{vmatrix} a_1(n+3) & a_2(n+3) \\ a_1(n+1) & a_2(n+1) \end{vmatrix} &= 0 \\ p_0(n)W_n(2,3) - p_2(n)W_{n+1}(2,3) + p_3(n)\begin{vmatrix} a_2(n+3) & a_3(n+3) \\ a_2(n+1) & a_3(n+1) \end{vmatrix} &= 0. \end{aligned}$$

- If we eliminate $p_2(n)$ and after some straightforward calculations we have

$$p_1(n)W_n(1,2,3) + p_3(n)W_{n+1}(1,2,3) = 0.$$

- Let us assume [2 equations, 4 polynomials]

$$
\begin{aligned}
p_0(n)a_1(n) + p_1(n)a_1(n+1) + p_2(n)a_1(n+2) + p_3(n)a_1(n+3) &= 0,\\
p_0(n)a_2(n) + p_1(n)a_2(n+1) + p_2(n)a_2(n+2) + p_3(n)a_2(n+3) &= 0.
\end{aligned}
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- If we work with such expressions in order to eliminate $p_1(n)$ we obtain

$$p_0(n)W_n(1,2) - p_2(n)W_{n+1}(1,2) + p_3(n)\begin{vmatrix} a_1(n+3) & a_2(n+3) \\ a_1(n+1) & a_2(n+1) \end{vmatrix} = 0.$$

Second case with Wronskians

- Let us assume [2 equations, 4 polynomials]

$$p_0(n)a_1(n) + p_1(n)a_1(n+1) + p_2(n)a_1(n+2) + p_3(n)a_1(n+3) = 0,$$
$$p_0(n)a_2(n) + p_1(n)a_2(n+1) + p_2(n)a_2(n+2) + p_3(n)a_2(n+3) = 0.$$

- If we work with such expressions in order to eliminate $p_1(n)$ we obtain

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- Taking into account the original recurrence by taking $n \mapsto n+1$, we multiply the previous expression by $p_0(n+1)$ and after some simplifications we have

$$p_0(n)p_0(n+1)W_n(1,2) - p_2(n)p_0(n+1)W_{n+1}(1,2)$$
$$+p_3(n)p_1(n+1)W_{n+2}(1,2) - p_3(n)p_3(n+1)W_{n+3}(1,2) = 0.$$

LATEST COMMENTS

- If $p_i(n) = \lambda_i n^T + \mathcal{O}(n^{T-1})$ $i = 0, 1, ..., 3$ and

$$p_0(n)u_1(n) + p_1(n)u_1(n+1) + p_2(n)u_1(n+2) + p_3(n)u_1(n+3) = 0$$

$$p_0(n)p_0(n+1)W_n(1,2) - p_2(n)p_0(n+1)W_{n+1}(1,2)$$
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- From the first expression we have

$$\lambda_0 + \lambda_1 x + \lambda_2 x^2 + \lambda_3 x^3 = 0$$

whose roots are r_1, r_2, r_3.

- If $p_i(n) = \lambda_i n^T + \mathcal{O}(n^{T-1})$ $i = 0, 1, \ldots, 3$ and

$$\begin{aligned} p_0(n)u_1(n) + p_1(n)u_1(n+1) + p_2(n)u_1(n+2) + p_3(n)u_1(n+3) &= 0 \\ p_0(n)p_0(n+1)W_n(1,2) - p_2(n)p_0(n+1)W_{n+1}(1,2) & \\ +p_3(n)p_1(n+1)W_{n+2}(1,2) - p_3(n)p_3(n+1)W_{n+3}(1,2) &= 0 \end{aligned}$$

- From the first expression we have

$$\lambda_0 + \lambda_1 x + \lambda_2 x^2 + \lambda_3 x^3 = 0$$

  whose roots are r_1, r_2, r_3.
- From the second equation we have

$$\lambda_3 + \lambda_2 \left(\frac{-\lambda_3}{\lambda_0} x\right) + \lambda_1 \left(\frac{-\lambda_3}{\lambda_0} x\right)^2 + \lambda_0 \left(\frac{-\lambda_3}{\lambda_0} x\right)^3 = 0$$

  whose roots are $r_1 r_2$, $r_1 r_3$, $r_2 r_3$.

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Thank you for your attention!

SLIDES: