2015 SIAM Orthogonal Polynomials, Special Functions and Applications

Generalizations of Generating Functions for Meixner and Krawtchouk Polynomial

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Outline

1. Define the Meixner and Krawtchouk Polynomials
2. Orthogonality of Meixner Polynomials in $\mathbb{C}$
3. Connection relations and connection-type relations for Meixner and Krawtchouk polynomials
4. Generalizations of generating functions for Meixner and Krawtchouk polynomials
5. Explanation how one may obtain orthogonality relations for these polynomials in $\mathbb{C}$ using Ramanujan's master theorem

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- **Generalized hypergeometric orthogonal polynomials**

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- **Basic hypergeometric orthogonal polynomials**

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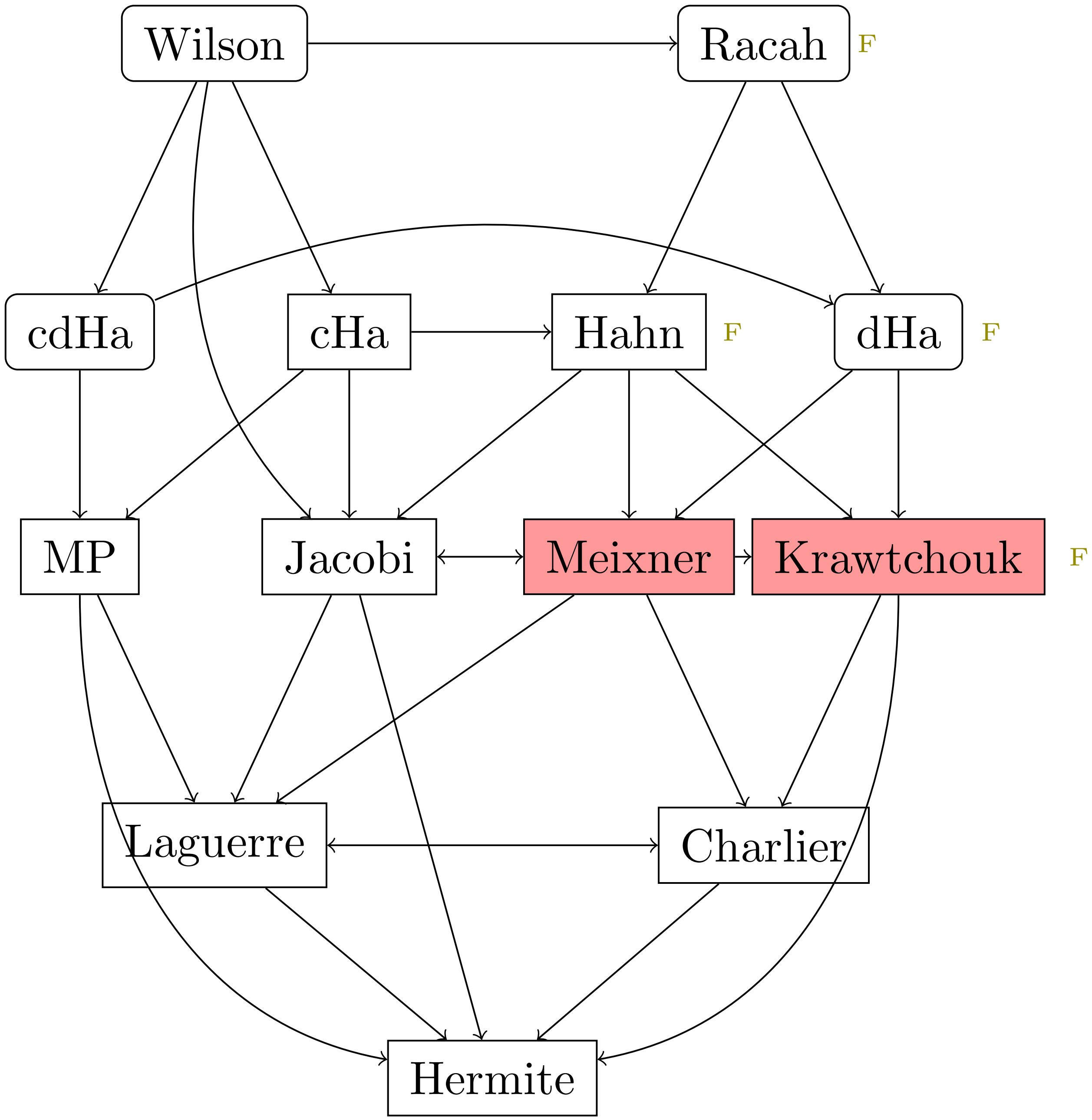
Hypergeometric Orthogonal Polynomials



Notation / definitions

Euler's gamma function and factorial for non-negative integers

$$\Gamma(z) := \int_0^\infty t^{z-1} e^{-t} dt, \quad \mathrm{Re}\, z > 0$$

Pochhammer symbol: the rising factorial in the complex plane

$$(a)_n := (a)(a+1)\dots(a+n-1), \quad (a)_0 := 1, \quad a \in \mathbb{C}$$

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$$

$$\Gamma(n+1) = n! = (1)_n$$

Generalized hypergeometric series

$${}_rF_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} ; z \right) := \sum_{n=0}^\infty \frac{(a_1)_n \dots (a_r)_n}{(b_1)_n \dots (b_s)_n} \frac{z^n}{n!}$$

1. The **Meixner** and Krawtchouk polynomials

The Meixner polynomials

$$M_n(x;\beta,c)=\frac{c^n(\beta)_n}{(c-1)^n}\,{}_2F_1\left(\begin{matrix}-n,-x\\ \beta\end{matrix};1-\frac{1}{c}\right)$$

The Krawtchouk polynomials

$$K_n(x;p,N)=(-N)_n p^n\,{}_2F_1\left(\begin{matrix}-n,-x\\ -N\end{matrix};\frac{1}{p}\right)$$

These two families of polynomials are related. Indeed,

$$K_n(x;p,N)=M_n\left(x;-N,\frac{p}{p-1}\right),\quad M_n(x;\beta,c)=K_n\left(x;-\beta,\frac{c}{c-1}\right)$$

2. The orthogonality for **Meixner** Polynomials

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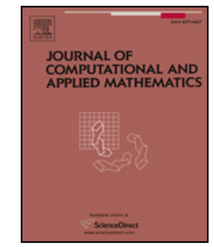

Extensions of discrete classical orthogonal polynomials beyond the orthogonality

R.S. Costas-Santos [a,*], J.F. Sánchez-Lara [b]

Proposition 9. *For any $\beta, c \in \mathbb{C}$, $c \notin [0, \infty)$ and $-\beta \notin \mathbb{N}$, the following property of orthogonality for the Meixner polynomials fulfills:*

$$\int_C M_n(z; c, \beta) z^m \Gamma(-z)\Gamma(\beta + z)(-c)^z \mathrm{d}z = 0, \quad 0 \le m < n, n = 0, 1, 2, \ldots \tag{A.3}$$

where C is a complex contour from $-\infty$i to ∞i separating the increasing poles $\{0, 1, 2, \ldots\}$ from the decreasing poles $\{-\beta, -\beta - 1, -\beta - 2, \ldots\}$.

Connection relations and coefficients

$$P_n^{(\alpha)}(x) = \sum_{k=0}^{n} c_{n,k}(\alpha;\beta) P_k^{(\beta)}(x)$$

What are the $c_{n,k}$? This is a **problem** in **orthogonal polynomials.** In general, one can compute connection relations by using **orthogonality**

$$\int_a^b P_k^{(\alpha)}(x) P_{k'}^{(\alpha)}(x) w(x;\alpha) dx = d_k(\alpha)\delta_{k,k'}.$$

Therefore

$$c_{n,k}(\alpha,\beta) = \frac{1}{d_k(\beta)} \int_a^b P_n^{(\alpha)}(x) P_k^{(\beta)}(x) w(x;\beta) dx.$$

Generating functions

$$f(x,\rho;\alpha) = \sum_{n=0}^{\infty} c_n(\alpha)\rho^n P_n^{(\alpha)}(x)$$

Examples:

- **Hermite polynomials**

$$\exp(2x\rho - \rho^2) = \sum_{n=0}^{\infty} \frac{1}{n!}\rho^n H_n(x)$$

- **Gegenbauer polynomials**

$$\frac{1}{(1+\rho^2-2\rho x)^{\nu}} = \sum_{n=0}^{\infty} \rho^n C_n^{\nu}(x)$$

- **Jacobi polynomials**

$$2^{\alpha+\beta}\mathsf{R}^{-1}(1-\rho+\mathsf{R})^{-\alpha}(1+\rho+\mathsf{R})^{-\beta} = \sum_{n=0}^{\infty} \rho^n P_n^{(\alpha,\beta)}(x),$$

where $\mathsf{R} = \sqrt{1+\rho^2-2\rho x}$.

Example: The Laguerre polynomials

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \frac{(n+\alpha)_{n-k}}{(n-k)!} \frac{(-x)^k}{k!} \qquad \textbf{(monic)}$$

Generating function (*Mourad's* trick)

$$(1-\rho)^{-\alpha-1} \exp\left(\frac{x\rho}{\rho-1}\right) = \sum_{n=0}^{\infty} \rho^n L_n^{(\alpha)}(x)$$

$$\frac{(1-\rho)^{-\alpha-1}}{(1-\rho)^{-\beta-1}}(1-\rho)^{-\beta-1} \exp\left(\frac{x\rho}{\rho-1}\right) = (1-\rho)^{\beta-\alpha} \sum_{n=0}^{\infty} \rho^n L_n^{(\beta)}(x)$$

$$(1-\rho)^{-r} = \sum_{k=0}^{\infty} \frac{(r)_k}{k!} x^{r-k} y^k \qquad \Longrightarrow \qquad (1-\rho)^{\beta-\alpha} = \sum_{j=0}^{\infty} \frac{(\alpha-\beta)_j}{j!} \rho^j$$

$$\sum_{j=0}^{\infty} \frac{(\alpha-\beta)_j}{j!} \rho^j \sum_{k=0}^{\infty} \rho^k L_k^{(\beta)}(x) = \sum_{n=0}^{\infty} \rho^n L_n^{(\alpha)}(x)$$

$$j+k=n \qquad \Longrightarrow \qquad j=n-k$$

$$\sum_{n=0}^{\infty} \left\{ L_n^{\alpha}(x) - \sum_{k=0}^{n} \frac{(\alpha-\beta)_{n-k}}{(n-k)!} L_k^{(\beta)}(x) \right\} \rho^n = 0$$

Connection relation (1 free parameter)

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} c_{n,k}(\alpha;\beta)\, L_k^{(\beta)}(x),$$

where

$$c_{n,k}(\alpha;\beta) = \frac{(\alpha-\beta)_{n-k}}{(n-k)!}$$

3a. Connection relations for **Meixner** polynomials

Theorem 1. *Let* $\alpha, \beta \in \widehat{\mathbb{C}},\ c, d \in \widehat{\mathbb{C}}_{0,1}$. *Then*

$$M_n(x;\alpha,c) = \sum_{k=0}^{n} \binom{n}{k} \frac{(\beta)_k}{(\alpha)_k} \left(\frac{d(1-c)}{c(1-d)}\right)^k {}_2F_1\left(\begin{matrix} -n+k, k+\beta \\ k+\alpha \end{matrix} ; \frac{d(1-c)}{c(1-d)}\right) M_k(x;\beta,d).$$

Corollary 2. *Let* $\alpha \in \widehat{\mathbb{C}},\ c, d \in \widehat{\mathbb{C}}_{0,1}$. *Then*

$$M_n(x;\alpha,c) = \left(\frac{c-d}{c(1-d)}\right)^n \sum_{k=0}^{n} \binom{n}{k} \left(\frac{d(1-c)}{c-d}\right)^k M_k(x;\alpha,d).$$

Corollary 3. *Let* $\alpha, \beta \in \widehat{\mathbb{C}},\ c \in \widehat{\mathbb{C}}_{0,1}$. *Then*

$$M_n(x;\alpha,c) = \frac{1}{(\alpha)_n} \sum_{k=0}^{n} \binom{n}{k} (\alpha-\beta)_{n-k} (\beta)_k M_k(x;\beta,c).$$

3b. Connection-type relations for **Meixner** polynomials

Theorem 4. *Let* $\alpha \in \widehat{\mathbb{C}}$, $c, d \in \widehat{\mathbb{C}}_{0,1}$. *Then*

$$M_n(x;\alpha,c) = \frac{1}{(\alpha)_n}\sum_{k=0}^{n}\binom{n}{k}\frac{(\alpha)_k(x)_{n-k}}{d^{n-k}}\,{}_2F_1\left(\begin{matrix}-n+k,-x\\ -x+k-n+1\end{matrix};\frac{d}{c}\right)M_k(x;\alpha,d).$$

Proof. A generating function for Meixner polynomials is given as

$$\left(1-\frac{t}{c}\right)^x(1-t)^{-x-\alpha} = \sum_{n=0}^{\infty}\frac{(\alpha)_n}{n!}M_n(x;\alpha,c)t^n, \qquad |t|<|c|<1.$$

The above connection-type relation (15) can be derived by starting with (16), and multiplying the left-hand side by $\left(1-\frac{t}{d}\right)^x \Big/ \left(1-\frac{t}{d}\right)^x$, $|t|<|d|<1$. One then has

$$\left(1-\frac{t}{c}\right)^x\left(1-\frac{t}{d}\right)^{-x}\left(1-\frac{t}{d}\right)^x(1-t)^{-x-\alpha} = \left(1-\frac{t}{c}\right)^x\left(1-\frac{t}{d}\right)^{-x}\sum_{m=0}^{\infty}\frac{(\alpha)_m}{m!}M_m(x;\alpha,d)t^m.$$

After using the binomial theorem (6), the left-hand side becomes

$$\sum_{k=0}^{\infty}\frac{(-x)_k}{k!}\left(\frac{t}{c}\right)^k\sum_{s=0}^{\infty}\frac{(x)_s}{s!}\left(\frac{t}{d}\right)^s\sum_{m=0}^{\infty}\frac{(\alpha)_m}{m!}M_m(x;\alpha,d)t^m.$$

By collecting the terms associated with t^n, (15) follows using analytic contination in c, d, and (2), (3) and (5). ■

3b. Connection-type relations for **Meixner** polynomials

Theorem 5. *Let* $\alpha, \beta \in \widehat{\mathbb{C}}$, $c, d \in \widehat{\mathbb{C}}_{0,1}$. *Then*

$$M_n(x;\alpha,c) = \frac{(\alpha-\beta)_n}{(\alpha)_n}\sum_{k=0}^{n}\frac{(\beta)_k(-n)_k}{k!(\beta-\alpha-n+1)_k}F_1\left(-n+k,-x,x;\beta-\alpha-n+k+1;\frac{1}{c},\frac{1}{d}\right)M_k(x;\beta,d). \tag{17}$$

The function F_1 is an Appell series, which are hypergeometric series in two variables and are defined as [4, (16.13.1)]

$$F_1\left(a,b,b';c;x,y\right) := \sum_{m,n=0}^{\infty}\frac{(a)_{m+n}(b)_m(b')_n}{(c)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!}. \tag{18}$$

3a. Connection relations for Krawtchouk polynomials

Theorem 6. *Let* $M, N \in \mathbb{N}_0$, $n \le N \le M$, $p, q \in \mathbb{C} \setminus \{0\}$. *Then*

$$K_n(x;p,N) = \sum_{k=0}^{n} \binom{n}{k} \frac{q^k(-M)_k}{p^k(-N)_k} {}_2F_1\left(\begin{matrix} -n+k, k-M \\ k-N \end{matrix} ; \frac{q}{p} \right) K_k(x;q,M).$$

Corollary 7. *Let* $p, q \in \mathbb{C} \setminus \{0\}$, $N \in \mathbb{N}_0$, $n \le N$. *Then*

$$K_n(x;p,N) = \left(\frac{p-q}{p}\right)^n \sum_{k=0}^{n} \binom{n}{k} \left(\frac{q}{p-q}\right)^k K_k(x;q,N).$$

Corollary 8. *Let* $p, q \in \mathbb{C} \setminus \{0\}$, $M, N \in \mathbb{N}_0$, $n \le N \le M$. *Then*

$$K_n(x;p,N) = \frac{1}{(-N)_n} \sum_{k=0}^{n} \binom{n}{k} (M-N)_{n-k}(-M)_k K_k(x;p,M).$$

4a. Generalizations of generating functions for Meixner polynomials

Theorem 9. *Let* $\alpha, \beta \in \widehat{\mathbb{C}}$, $c, d \in \widehat{\mathbb{C}}_{0,1}$, $t \in \mathbb{C}$. *Then*

$$e^t {}_1F_1\left(\begin{matrix}-x \\ \alpha\end{matrix}; \frac{t(1-c)}{c}\right) = \sum_{n=0}^{\infty} \frac{(\beta)_n}{(\alpha)_n n!} \left(\frac{d(1-c)}{c(1-d)}\right)^n {}_1F_1\left(\begin{matrix}\beta+n \\ \alpha+n\end{matrix}; \frac{-td(1-c)}{c(1-d)}\right) M_n(x;\beta,d) t^n.$$

Proof. Using the generating function for Meixner polynomials [6, (9.10.12)]

$$e^t {}_1F_1\left(\begin{matrix}-x \\ \alpha\end{matrix}; \frac{t(1-c)}{c}\right) = \sum_{n=0}^{\infty} \frac{t^n}{n!} M_k(x;\beta,d)$$

and (12), we obtain

$$e^t {}_1F_1\left(\begin{matrix}-x \\ \alpha\end{matrix}; \frac{t(1-c)}{c}\right) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{k=0}^{n} \binom{n}{k} \frac{(\beta)_k}{(\alpha)_k} \left(\frac{d(1-c)}{c(1-d)}\right)^k {}_2F_1\left(\begin{matrix}-n+k, \beta+k \\ \alpha+k\end{matrix}; \frac{d(1-c)}{c(1-d)}\right) M_k(x;\beta,d).$$

Let $j \in \mathbb{N}$, $k, n \in \mathbb{N}_0$, $z \in \mathbb{C}$, $\Re u > 0$, $w > -1$, $v \geq 0$. Then

$$|(u)_j| \geq (\Re u)(j-1)!,$$

$$\frac{(v)_n}{n!} \leq (1+n)^v,$$

$$(n+w)_k \leq \max\{1, 2^w\} \frac{(n+k)!}{n!},$$

$$(z+k)_{n-k} \leq \frac{n!}{k!}(1+n)^{|z|}.$$

4a. Generalizations of generating functions for Meixner polynomials

Theorem 10. *Let* $\alpha, \beta \in \widehat{\mathbb{C}}$, $c \in \widehat{\mathbb{C}}_{0,1}$, $t \in \mathbb{C}$. *Then*

$$e^t {}_1F_1\left(\begin{matrix} -x \\ \alpha \end{matrix}; \frac{t(1-c)}{c}\right) = \sum_{n=0}^{\infty} \frac{(\beta)_n}{(\alpha)_n n!} {}_1F_1\left(\begin{matrix} \alpha-\beta \\ \alpha+n \end{matrix}; t\right) M_n(x;\beta,c) t^n.$$

Theorem 11. *Let* $\alpha \in \widehat{\mathbb{C}}$, $c, d \in \widehat{\mathbb{C}}_{0,1}$, $t \in \mathbb{C}$. *Then*

$$e^t {}_1F_1\left(\begin{matrix} -x \\ \alpha \end{matrix}; \frac{t(1-c)}{c}\right) = \sum_{n=0}^{\infty} \frac{1}{n!} \Phi_2\left(x, -x; \alpha+n; \frac{t}{c}, \frac{t}{d}\right) M_n(x;\alpha,d) t^n.$$

The function Φ_2 is a Humbert hypergeometric series of two variables defined as

$$\Phi_2\left(\beta, \beta'; \gamma; x, y\right) := \sum_{m,n=0}^{\infty} \frac{(\beta)_m (\beta')_n}{(\gamma)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!}.$$

Theorem 12. *Let* $\alpha, \beta \in \widehat{\mathbb{C}}$, $c, d \in \widehat{\mathbb{C}}_{0,1}$, $t \in \mathbb{C}$. *Then*

$$e^t {}_1F_1\left(\begin{matrix} -x \\ \alpha \end{matrix}; \frac{t(1-c)}{c}\right) = \sum_{n=0}^{\infty} \frac{(\beta)_n}{(\alpha)_n n!} \Phi_2^{(3)}\left(x, -x, \alpha-\beta; \alpha+n; \frac{t}{c}, \frac{t}{d}, t\right) M_n(x;\beta,d) t^n.$$

The function $\Phi_2^{(3)}$ is a confluent form of the Lauricella series defined as [7, p. 34]

$$\Phi_2^{(3)}\left(b_1, b_2, b_3; c; x_1, x_2, x_3\right) := \sum_{m_1,m_2,m_3=0}^{\infty} \frac{(b_1)_{m_1} (b_2)_{m_2} (b_3)_{m_3}}{(c)_{m_1+m_2+m_3}} \frac{x_1^{m_1}}{m_1!} \frac{x_2^{m_2}}{m_2!} \frac{x_3^{m_3}}{m_3!}.$$

4a. Generalizations of generating functions for Meixner polynomials

Theorem 13. *Let* $|t| < 1$, $|t(1-c)| < |c(1-t)|$, $\alpha, \beta \in \widehat{\mathbb{C}}$, $\gamma \in \mathbb{C}$, $c \in \widehat{\mathbb{C}}_{0,1}$. *Then*

$$(1-t)^{-\gamma}{}_2F_1\left(\begin{matrix}\gamma, -x \\ \alpha\end{matrix}; \frac{t(1-c)}{c(1-t)}\right) = \sum_{n=0}^{\infty} \frac{(\gamma)_n(\beta)_n}{(\alpha)_n n!}{}_2F_1\left(\begin{matrix}\gamma+n, \alpha-\beta \\ \alpha+n\end{matrix}; t\right) M_n(x;\beta,c)t^n.$$

Theorem 14. *Let* $|t| < \min\{1, |c(1-d)|/|1+d-2c|\}$, $\alpha, \beta \in \widehat{\mathbb{C}}$, $\gamma \in \mathbb{C}$, $c \in \widehat{\mathbb{C}}_{0,1}$. *Then*

$$(1-t)^{-\gamma}{}_2F_1\left(\begin{matrix}\gamma, -x \\ \alpha\end{matrix}; \frac{t(1-c)}{c(1-t)}\right) = \sum_{n=0}^{\infty} \frac{(\gamma)_n(\beta)_n}{(\alpha)_n n!}{}_2F_1\left(\begin{matrix}\gamma+n, \beta+n \\ \alpha+n\end{matrix}; \frac{-dt(1-c)}{c(1-d)(1-t)}\right) \times \left(\frac{d(1-c)}{c(1-d)(1-t)}\right)^n M_n(x;\beta,d)t^n.$$

There are much more … soon in Arxiv!

5. The orthogonality & the Ramanujan's Master Theorem

Let $\mathbf{A}$, $\mathbf{P}$, δ be real constants so that $\mathbf{A} < \pi$ and $0 < \delta \leq 1$. Let $\mathcal{H}(\delta) = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > -\delta\}$. The Hardy class $\mathcal{H}(\mathbf{A}, \mathbf{P}, \delta)$ consists of all functions $a : \mathcal{H}(\delta) \to \mathbb{C}$ that are holomorphic on $\mathcal{H}(\delta)$ and satisfy the growth condition

$$|a(\lambda)| \leq Ce^{-\mathbf{P}(\operatorname{Re}\lambda)+\mathbf{A}|\operatorname{Im}\lambda|}$$

for all $\lambda \in \mathcal{H}(\delta)$. Hardy's version of Ramanujan's Master theorem is the following, see [16,

Theorem 0.1 (Ramanujan's Master Theorem). *Suppose $a \in \mathcal{H}(\mathbf{A}, \mathbf{P}, \delta)$. Then:*

(a) *The power series*

$$f(x) = \sum_{k=0}^{\infty} (-1)^k a(k) x^k \tag{0.3}$$

converges for $0 < x < e^{\mathbf{P}}$ and defines a real analytic function on this domain.

(b) *Let $0 < \sigma < \delta$. For $0 < x < e^{\mathbf{P}}$ we have*

$$f(x) = \frac{1}{2\pi i} \int_{-\sigma - i\infty}^{-\sigma + i\infty} \frac{-\pi}{\sin(\pi\lambda)}\, a(\lambda) x^{\lambda}\, d\lambda\,. \tag{0.4}$$

The integral on the right hand side of (0.4) converges uniformly on compact subsets of $]0, +\infty[$ and is independent of the choice of σ.

(c) *Formula (0.1) holds for the extension of f to $]0, +\infty[$ and for all $\lambda \in \mathbb{C}$ with $0 < \operatorname{Re}\lambda < \delta$.*

The orthogonality for Meixner polynomials

We choose $\sigma = 1/2$, $x = -c$, and since

$$\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin(\pi z)} \quad \Rightarrow \quad \Gamma(1+z)\Gamma(-z) = \frac{-\pi}{\sin(\pi z)}$$

then choosing

$$a(z) = \frac{\Gamma(\beta+z)}{\Gamma(z+1)} M_n(z;c,\beta) z^m, \quad \beta > 0,$$

we get

$$\int_C \Gamma(-z)\Gamma(\beta+z)(-c)^z M_n(z;c,\beta) z^m \, dz = \sum_{k=0}^{\infty} \frac{\Gamma(\beta+k)}{\Gamma(k+1)} c^k M_n(k;c,\beta) z^m.$$

References

1. arXiv:1411.1371 [pdf, ps, other]
Generalizations of generating functions for basic hypergeometric orthogonal polynomials
Howard S. Cohl, Roberto S. Costas-Santos, Philbert R. Hwang
Subjects: **Classical Analysis and ODEs (math.CA)**

Maximal Meixner generalized generating functions and connection-type relations

Michael A. Baeder, Howard S. Cohl,†, Roberto S. Costas-Santos,‡ and Wenqing Xu§*

and few more references by Howard Cohl et al.


obrigado
Dank U
Merci
mahalo
Köszi
спасибо
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Thank you
mauruuru
Takk
Gracias
Dziękuję
Děkuju
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Kiitos