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On a degenerate version of Favard's theorem

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Outline

1. The Favard’s theorem
2. Degenerate version of Favard’s theorem
3. An example: The Askey-Wilson Polynomials

The Favard's theorem

TTRR

- Consider the polynomials $(p_n)_{n\in\mathbb{N}_0}$ generated by the TTRR

$$xp_n(x) = p_{n+1}(x) + \beta_n p_n(x) + \gamma_n p_{n-1}(x),$$

with initial conditions $p_{-1}(x) \equiv 0$, $p_0(x) = 1$.

Theorem (Favard) If $\gamma_n \neq 0\ \forall n \in \mathbb{N}$ then there exists a linear $\mathscr{L}_0 : \mathbb{P}[x] \to \mathbb{C}$ so that

$$\mathscr{L}_0(p_n p_m) = r_n \delta_{n,m}$$

(r_n a non-vanishing normalization factor)

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$$\mathscr{L}_0(p_n) := \delta_{0,n}$$

Proof of Favard's theorem

Hypothesis: $xp_n(x) = p_{n+1}(x) + \beta_n p_n(x) + \gamma_n p_{n-1}(x)$,
$\mathscr{L}_0(p_n) := \delta_{n,0}, \qquad \gamma_n \neq 0$ for all $n \in \mathbb{N}$.

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- $\mathscr{L}_0(p_N p_N) = 0$

Orthogonality with respect to $\mathscr{L}_0$ Polynomials are orthogonal with respect to $\mathscr{L}_0$ does not characterize the polynomials since $p_{N+1} + \alpha p_N \perp \langle p_0, \ldots, p_N \rangle$.

Degenerate version of Favard's theorem

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Many times we get $\gamma_{n,1} = 0 \iff \gamma_{n+1} = 0$

Consequence: $(p_{n,1})$ are orthogonal with respect to some $\mathscr{L}_1$, and the first n such that $\gamma_{n,1} = 0$ verifies $n < N$.

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Theorem: Suppose that only $\gamma_N = 0$, then (p_n) is a MOPS with respect to

$$\langle f, g \rangle = \mathscr{L}_0(fg) + \mathscr{L}_N(\mathscr{T}^{(N)}(f)\mathscr{T}^{(N)}(g)).$$

Notice $\gamma_{n,N} \neq 0$ for all $n \in \mathbb{N}$

The proof

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- If $n \geq N$ then

$$\begin{aligned}\langle p_n, p_n\rangle = \quad & \text{const.}\mathscr{L}_N(p_{n-N}p_{n-N}) = \\ & \text{const.}\gamma_{n-N,N}\gamma_{n-N-1,N}\cdots\gamma_{1,N} \neq 0.\end{aligned}$$

Corollary

Corollary: If $\Lambda = \{n : \gamma_n = 0\}$, then (p_n) is a MOPS with respect to

$$\langle f, g\rangle = \mathscr{L}_0(fg) + \sum_{j\in\mathscr{A}} \mathscr{L}_j(\mathscr{T}^{(j)}(f)\mathscr{T}^{(j)}(g)),$$

being $\mathscr{A} = \{N_0, N_1, \dots\}$ with
$N_{j+1} = N_j + \min\{n : \gamma_{n,N_j} = 0\}$.

The operator $\mathscr{T}$

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- The "Associating operator"

$$\mathscr{T}(p)(x) = \mathscr{L}_0\left(\frac{p(x)-p(t)}{x-t}\right)$$

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- If (p_n) is classical, then $\mathscr{T}$ is
  - the derivative, or
  - a difference operator.

Previous results

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- $\mathscr{T}$ a difference operator:
  - R. Costas-Santos, J. Sánchez-Lara (2009): Racah, Hahn, Dual Hahn, Krawtchouk

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- $\mathscr{T}$ the q-difference operator:
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  - R. Costas-Santos, J. Sánchez-Lara (2010):

Askey-Wilson, Big q-Jacobi, q-Racah, q-Hahn, etc

The example: Askey-Wilson polynomials

Monic Askey-Wilson polynomials

The monic ones are $p_n(x; a, b, c, d; q) \equiv p_n(x)$

$$p_{n+1}(x) = (x - \beta_n)p_n(x) - \gamma_n p_n(x),$$

with

$$\frac{\gamma_n}{1-q^n} = \frac{(1-abq^{n-1})(1-acq^{n-1})(1-adq^{n-1})(1-bcq^{n-1})(1-bdq^{n-1})(1-cdq^{n-1})}{4(1-abcdq^{2n-3})(1-abcdq^{2n-2})^2(1-abcdq^{2n-1})}$$

Case $abcd \in \{q^{-k} : k \in \mathbb{N}_0\}$ are not considered since they are not normal.
They are symmetric with respect to any rearrangement of the parameters a, b, c, d.

$$\{n \in \mathbb{N} : \gamma_n = 0\} \neq \emptyset \iff ab, ac, \ldots, cd \in \{q^{-k} : k \in \mathbb{N}_0\}$$

$\iff$ they are q-Racah (until now considered as a finite family).

Orthogonality of AW polynomials for $|q| < 1$

$$\int_C p_n\left(\frac{z+z^{-1}}{2}\right) p_m\left(\frac{z+z^{-1}}{2}\right) W(z)dz = d_n\delta_{n,m}$$

where

- W is analytic in $\mathbb{C}$ except at the poles 0,

$$aq^k, bq^k, cq^k, dq^k \quad k \in \mathbb{N}_0 \quad \text{(the convergent poles)}$$

$$(aq)^{-k}, (bq)^{-k}, (cq)^{-k}, (dq)^{-k} \quad k \in \mathbb{N}_0 \quad \text{(the divergent poles)}$$

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- $a^2, b^2, c^2, d^2 \notin \{q^{-k} : k \in \mathbb{N}_0\}$ ($\gamma_n \neq 0$ for all n)
- $ab, ac, ad, bc, bd, cd \in \{q^{-k} : k \in \mathbb{N}_0\}$ (some $\gamma_N = 0$)

The 3 key cases

- Case I: $a^2 = q^{-N+1}$ and

$$b^2, c^2, d^2, ab, ac, ad, bc, bd, cd \notin \{q^{-k} : k \in \mathbb{N}_0\}$$

- Case II: $ab = q^{-N+1}$ and

$$a^2, b^2, c^2, d^2, ac, ad, bc, bd, cd \notin \{q^{-k} : k \in \mathbb{N}_0\}$$

- Case III: $ab = q^{-N+1}$, $a^2 = q^{-M}$ with $M \in \{0, 1, \dots, N-2\}$ and

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Poles:

$$\ldots, \pm q^{-M/2-1}, \pm q^{-M/2}, \pm q^{-M/2+1}, \ldots, \pm q^{M/2-1}, \pm q^{M/2}, \pm q^{M/2+1}, \ldots$$

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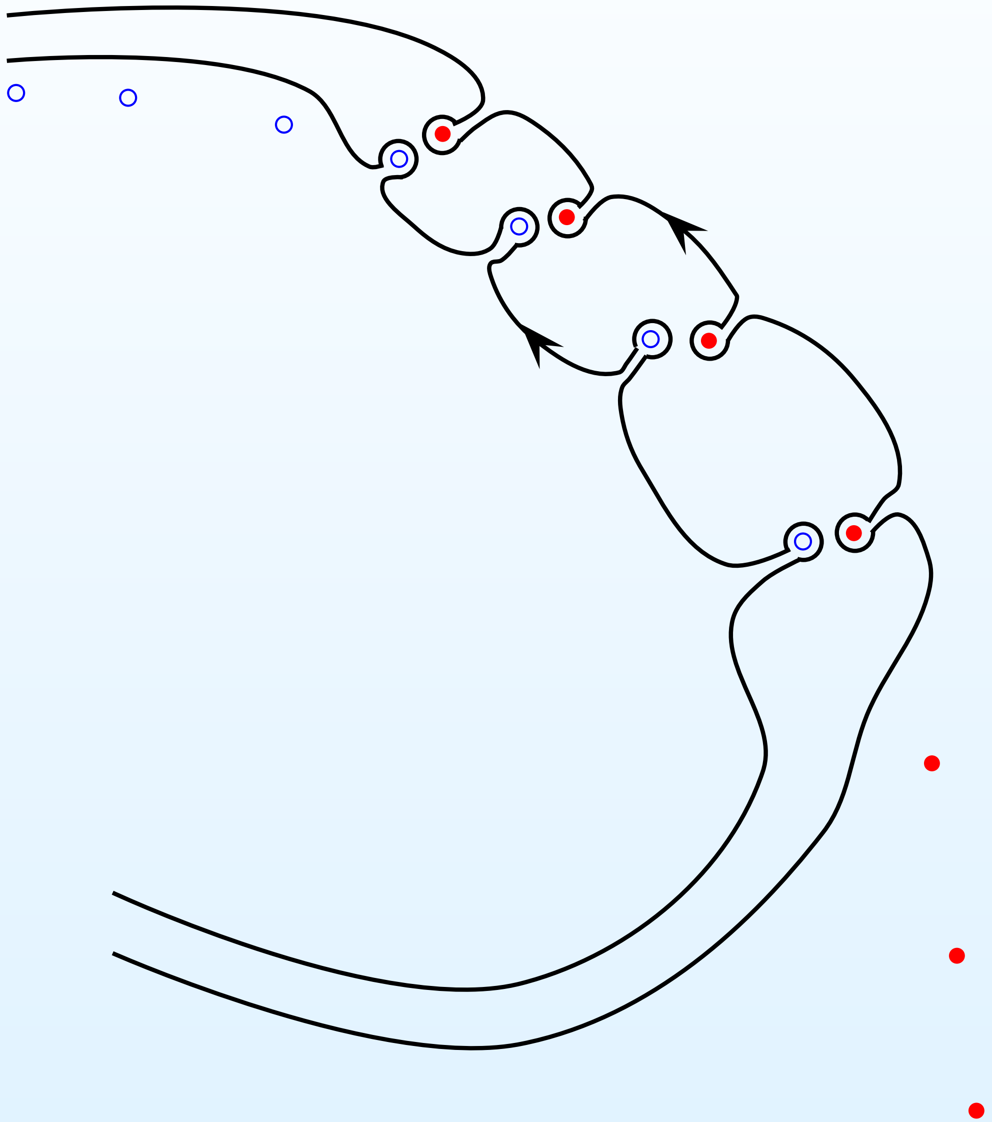

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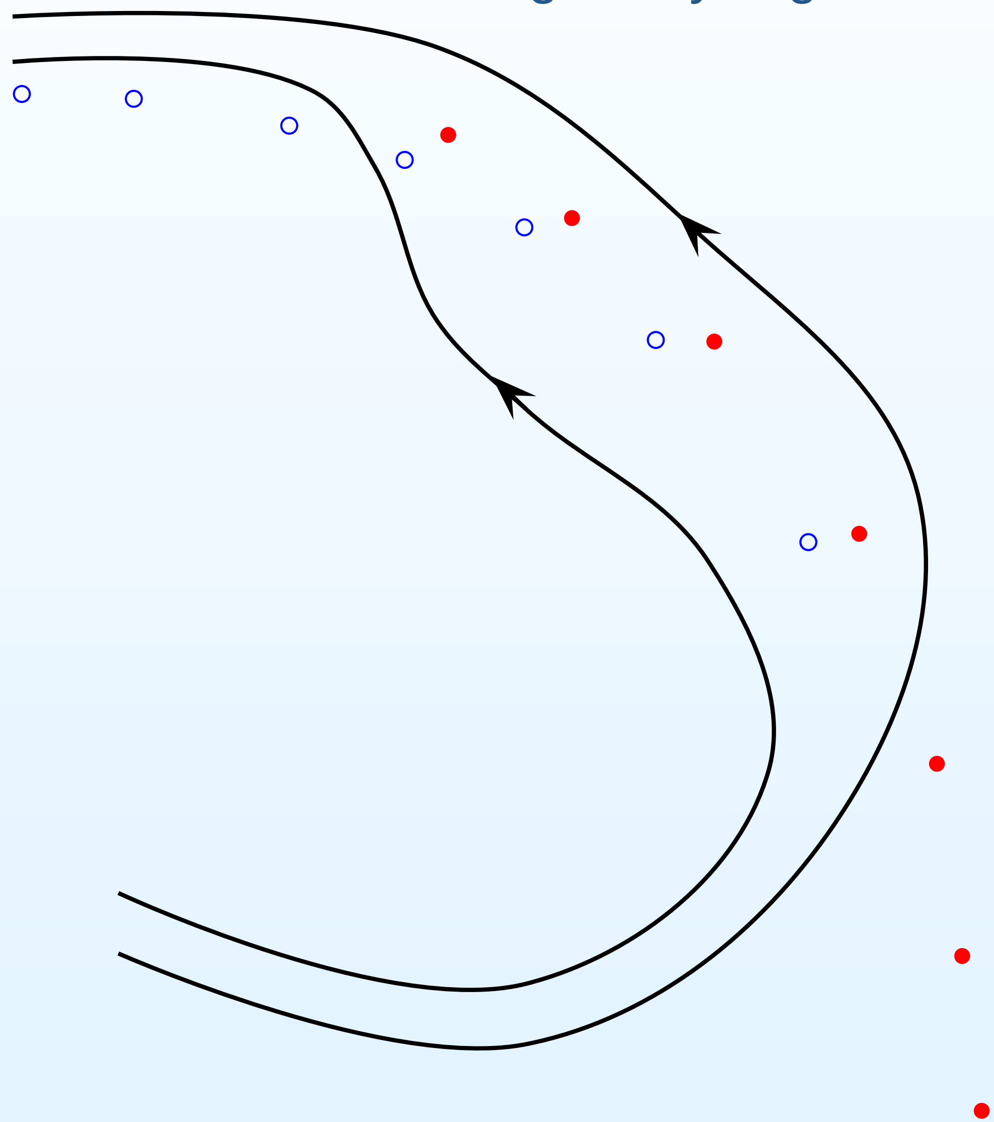

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$$\mathscr{L}_0(p; a, b, c, d) = \left(\int_{C_1} + \int_{C_2} \right) p(z) W(z) dz$$

with C_1 and C_2 separating the divergent poles from the convergent ones but the double poles which stand between C_1 and C_2.

Case II: $ab = q^{-N+1}$

In this case $\gamma_N = 0$ (the unique)$\Rightarrow$ we need $\mathscr{L}_0$, $\mathscr{L}_N$.

- $\mathscr{L}_0$ is a quadrature rule.

These AW polynomials are the q-Racah polynomials

$$\mathscr{L}_0(p) = \sum_{j=0}^{N-1} \frac{(q^{-N+1}, ac, ad, a^2; q)_j}{(q, a^2q^N, ac^{-1}q, ad^{-1}q; q)_j} \frac{(1-a^2q^{2j})}{(cdq^{-N})^j(1-a^2)} p\left(\frac{q^{-j}+a^2q^{2j}}{2a}\right)$$

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- $\mathscr{T} = \mathscr{D}_q$ the Hahn's operator

$$\mathscr{D}_q(f)(z) \stackrel{\text{def}}{=} \begin{cases} \dfrac{f(z)-f(qz)}{(1-q)z}, & z \neq 0 \wedge q \neq 1, \\ f'(z), & z = 0 \vee q = 1, \end{cases}$$

$$\mathscr{D}^N p_n(x; a, b, c, d; q) = \text{const.} p_{n-N}(x; aq^{N/2}, bq^{N/2}, cq^{N/2}, dq^{N/2}; q)$$

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- $\mathscr{L}_N(p; a, b, c, d) = \mathscr{L}_0(p; aq^{N/2}, bq^{N/2}, cq^{N/2}, dq^{N/2})$

Case III

$ab = q^{-N+1}$ and $a^2 = q^{-M}$, with $M \in \{0, \ldots, N-2\}$ with only $\gamma_N = 0 \Rightarrow$ we need $\mathscr{L}_0$, $\mathscr{L}_N$.

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The result is a quadrature rule with simple and double nodes

Orthogonality of AW polynomials for $|q| \geq 1$

- $|q| > 1.$

$$p_n(x; a, b, c, d|q^{-1}) = p_n(x; a^{-1}, b^{-1}, c^{-1}, d^{-1}|q)$$

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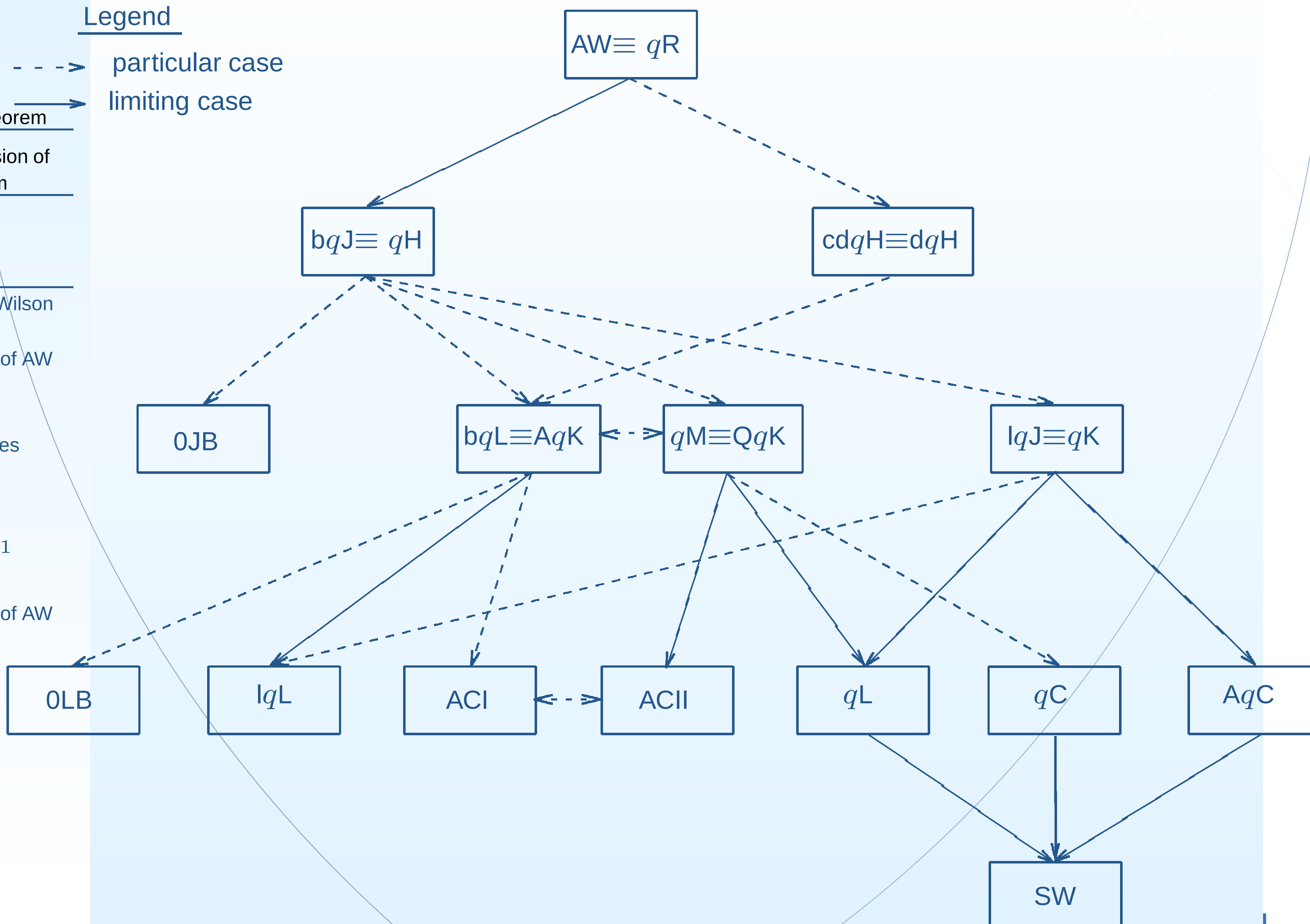
- For the rest of the values of q the result keeps UNKNOWN.

The scheme



Legend
particular case
limiting case
AW≡ qR
bqJ≡ qH
cdqH≡dqH
0JB
bqL≡AqK
qM≡QqK
lqJ≡qK
0LB
lqL
ACI
ACII
qL
qC
AqC
SW

Finally ....

THANK YOU FOR YOUR ATTENTION !!