Article

On polar Jacobi polynomials

Roberto S. Costas-Santos [1]

[1] Department of Quantitative Methods, Universidad Loyola Andalucía, E-41704 Seville, Spain; rscosa@gmail.com

Abstract: In this paper, we study some algebraic, differential, and asymptotic properties of the orthogonal polynomials with respect to discrete-continuous Sobolev-type inner product with respect to the Jacobi measure.

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1. Introduction

Let us consider a measure μ supported on the subset of the complex plane γ. In the vector space $\mathbb{P}$ of all polynomials with complex coefficients, we define the inner product

$$\langle f, g\rangle_\xi := Mf(\xi)g(\xi) + \int_\gamma f'(z)g'(z)d\mu(z), \quad M, \xi \in \mathbb{C}, \tag{1}$$

assuming that the integral exists. The inner product (1) is called a discrete-continuous Sobolev-type, which is a particular case of the Sobolev-type inner products. Algebraic and analytical properties and the asymptotic behavior of the families of orthogonal polynomials with respect to Sobolev-type inner products have been extensively used for the last 25 years. For an overview of this subject, see [17], or the introduction of [14] as well as the [16].

The discrete-continuous Sobolev inner products were introduced in [9] to study the behavior of the best polynomial approximation of absolutely continuous functions in the norm generated by a Sobolev inner product as (1). Later, in [11], R. Koekoek considered the Laguerre case with $d\mu = x^\alpha e^{-x}dx$, $\alpha > -1$, $\gamma = [0, +\infty)$ and $\xi = 0$. The Gegenbauer case was studied by Bavinck and Meijer in [3,4] with $d\mu = (1-x^2)^{\lambda-1/2}dx$, $\lambda > -1/2$, $\gamma = [-1, 1]$ and $\xi_1 = -1$ and $\xi_2 = 1$.

The families of polynomials orthogonal with respect to this type of inner products have been studied as the extension of the Bochner-Krall theory (i.e., families of polynomials that are simultaneously eigenfunctions of a differential operator and orthogonal with respect to an inner product, see for the discrete-continuous [1,10,13]

The starting point of our work is the orthogonality with respect to the Jacobi case. Let (Q_n) be the family of monic orthogonal polynomials with respect to the Sobolev inner product

$$\langle f, g\rangle := f(\xi)g(\xi) + \int_\gamma f'(z)g'(z)(1-z)^\alpha(1+z)^\beta dz, \tag{2}$$

where $\alpha, \beta \in \mathbb{C}$, γ is a path encircling the points $+1$ and -1 first in a positive sense and then in a negative sense, as shown in [12, Figure 2.1], $M = 1$, and $\xi \in \mathbb{C}$.

Let us consider $\left(P_n^{(\alpha,\beta)}(z)\right)$ be the monic Jacobi polynomials, i.e., the family of monic orthogonal polynomials with respect to the measure μ. Therefore, for each n, $P_n^{(\alpha,\beta)}(z)$ satisfies

$$\int_\gamma P_n^{(\alpha,\beta)}(z) z^k (1-z)^\alpha (1+z)^\beta dz = 0, \quad k = 0, 1, ..., n-1, \tag{3}$$

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and let $\Pi_{n+1,\xi}$ be the polynomial primitive of $(n+1)P_n^{(\alpha,\beta)}$ vanishing at ξ, i.e. for all $n \geq 1$ we have

$$\Pi_{n+1,\xi}(\xi) = 0, \qquad \frac{d}{dz}\Pi_{n+1,\xi}(z) = (n+1)P_n^{(\alpha,\beta)}(z). \tag{4}$$

Then, by definition of Q_n, it is clear that $\Pi_{n+1,\xi}(z) = Q_{n+1}(z)$ for all $n = 0, 1, 2, ...$ It is straightforward to prove that the n-th orthogonal polynomial with respect to the Sobolev inner product (2) can be written as

$$Q_n(z) = (z-\xi)P_{n-1}(z),$$

where P_n is called the `polar polynomial associated with` μ (see [18]) and ξ from now on will be called the `pole`. Let us define the differential operator $L_\xi : \mathcal{H}^1(\gamma) \to L^2(\gamma)$ as

$$L_\xi[f(z)] = f(z) + (z-\xi)\frac{d}{dz}f(z), \tag{5}$$

where $\mathcal{H}^1(\gamma) := \{f \in L^2(\gamma) : f'(z) \in L^2(\gamma)\}$ is the Sobolev space of index 1. Take into account Q_n is orthogonal with respect to the inner product (2), we have

$$\int_\gamma L_\xi[P_n(z)]z^k(1-z)^\alpha(1+z)^\beta dz = \int_\gamma \left(P_n(z) + (z-\xi)P_n'(z)\right)z^k(1-z)^\alpha(1+z)^\beta dz = 0,$$

for $k = 0, 1, ..., n-1$. Therefore, P_n is the nth monic orthogonal polynomial with respect to the differential operator L_ξ, and the measure

$$d\mu = \omega(z;\alpha,\beta)dz := (1-z)^\alpha(1+z)^\beta dz,$$

see [2,5–7,18,19]. In such a case, we have

$$P_n(z) + (z-\xi)P_n'(z) = (n+1)P_n^{(\alpha,\beta)}(z). \tag{6}$$

The main goal of this article is to study algebraic (zero localization), differential, and asymptotic properties of the orthogonal polynomials with respect to the inner product (2) for the Jacobi case, which is a natural extension of the Legendre case [18].

In Section 2 we obtain several algebraic relations between the polar Jacobi polynomials and the Jacobi polynomials and some differential and different identities related to the polar Jacobi polynomials. In Section 3 we study the location of the zeros for the polynomials P_n. Finally, in Section 4 we discus some possible extensions of the results

2. Algebraic properties of the polar Jacobi polynomials

Let us start by summarizing some basic properties of the Jacobi orthogonal polynomials to be used in the sequel.

Proposition 1. *Let* $\left(P_n^{(\alpha,\beta)}(z)\right)$ *be the classical monic Jacobi orthogonal polynomials sequence. The following statements hold:*

1. *Three-term recurrence relation.*

$$P_{n+1}^{(\alpha,\beta)}(z) = (z-\beta_n)P_n^{(\alpha,\beta)}(z) - \gamma_n P_{n-1}^{(\alpha,\beta)}(z), \quad n = 0, 1, ..., \tag{7}$$

with initial condition $P_0^{(\alpha,\beta)}(z) = 1$*, and recurrence coefficients*

$$\beta_n = \frac{\beta^2-\alpha^2}{(\alpha+\beta+2n)(\alpha+\beta+2n+2)}, \quad \gamma_n = \frac{4n(\alpha+n)(\beta+n)(\alpha+\beta+n)}{(\alpha+\beta+2n-1)(\alpha+\beta+2n)^2(\alpha+\beta+2n+1)}.$$

2. *First structure relation.*

$$(1-z^2)\frac{d}{dz}P_n^{(\alpha,\beta)}(z) = -nP_{n+1}^{(\alpha,\beta)}(z) + \widehat{\beta}_n P_n^{(\alpha,\beta)}(z) + \widehat{\gamma}_n P_{n-1}^{(\alpha,\beta)}(z), \quad n = 0,1,...., \tag{8}$$

with coefficients

$$\widehat{\beta}_n = \frac{2n(\alpha-\beta)(\alpha+\beta+n+1)}{(\alpha+\beta+2n)(\alpha+\beta+2n+2)}, \quad \widehat{\gamma}_n = \frac{4n(\alpha+n)(\beta+n)(\alpha+\beta+n)(\alpha+\beta+n+1)}{(\alpha+\beta+2n-1)(\alpha+\beta+2n)^2(\alpha+\beta+2n+1)}.$$

3. *Second structure relation.*

$$P_{n+1}^{(\alpha-1,\beta-1)}(z) = P_{n+1}^{(\alpha,\beta)}(z) + \widetilde{\beta}_n P_n^{(\alpha,\beta)}(z) + \widetilde{\gamma}_n P_{n-1}^{(\alpha,\beta)}(z), \quad n = 0,1,...., \tag{9}$$

$$\widetilde{\beta}_n = \frac{(2n+2)(\alpha-\beta)}{(\alpha+\beta+2n)(\alpha+\beta+2n+2)}, \quad \widetilde{\gamma}_n = -\frac{4n(n+1)(\alpha+n)(\beta+n)}{(\alpha+\beta+2n-1)(\alpha+\beta+2n)^2(\alpha+\beta+2n+1)}$$

4. *Squared Norm. For every $n \geq 0$,*

$$\left\| P_n^{(\alpha,\beta)}(z) \right\|^2 = \int_\gamma \left(P_n^{(\alpha,\beta)}(z) \right)^2 \omega(z;\alpha,\beta)\,dz = \frac{4^n n!\Gamma(\alpha+n+1)\Gamma(\beta+n+1)\Gamma(\alpha+\beta+n+1)}{\Gamma(\alpha+\beta+2n+1)\Gamma(\alpha+\beta+2n+2)}. \tag{10}$$

5. *Second-order difference equation. For every $n \geq 0$,*

$$(1-z^2)\frac{d^2}{dz^2}P_n^{(\alpha,\beta)}(z) + (\beta-\alpha-z(\alpha+\beta))\frac{d}{dz}P_n^{(\alpha,\beta)}(z) = -n(\alpha+\beta+n+1)P_n^{(\alpha,\beta)}(z). \tag{11}$$

6. *Forward shift operator.*

$$\frac{d}{dz}P_n^{(\alpha,\beta)}(z) = nP_{n-1}^{(\alpha+1,\beta+1)}(z), \quad n = 0,1,...., \tag{12}$$

7. *Asymptotic formula. Let $z \in \mathbb{C} \setminus [-1,1]$. Put $\varphi(z) = z + \sqrt{z^2-1}$ where the branch of the square root is chosen so that $|z + \sqrt{z^2-1}| > 1$ for $z \in \mathbb{C} \setminus [-1,1]$. Then*

$$P_n^{(\alpha,\beta)}(z) = \frac{\varphi^n(z)}{\sqrt{n}}\Big(c(\alpha,\beta,z) + \mathcal{O}(n^{-1})\Big), \tag{13}$$

where $c(\alpha,\beta,z)$ is a function of α, β and x independent of n. The relation holds uniformly on compact sets of $\mathbb{C} \setminus [-1,1]$.

Let us obtain the algebraic relations between the Jacobi polynomials and the polar Jacobi polynomials.

Lemma 1. *For any $\alpha, \beta, \xi \in \mathbb{C}$. The polar Jacobi polynomials can be written in terms of the Jacobi polynomials as follows:*

$$\begin{aligned} P_n(z) &= \frac{P_{n+1}^{(\alpha-1,\beta-1)}(z) - P_{n+1}^{(\alpha-1,\beta-1)}(\xi)}{z-\xi} && (14) \\ &= \frac{1}{\alpha+\beta+n}\Bigg[(z+\xi)\frac{d}{dz}P_n^{(\alpha,\beta)}(z) + \frac{\xi^2-1}{z-\xi}\left(\frac{d}{dz}P_n^{(\alpha,\beta)}(z) - \frac{d}{dz}P_n^{(\alpha,\beta)}(\xi)\right) \\ &\quad +(\alpha+\beta)P_n^{(\alpha,\beta)}(z) + \frac{\alpha-\beta+\xi(\alpha+\beta)}{z-\xi}\Big(P_n^{(\alpha,\beta)}(z) - P_n^{(\alpha,\beta)}(\xi)\Big)\Bigg]. && (15) \end{aligned}$$

Therefore

$$(z-\xi)P_n(z) = P_{n+1}^{(\alpha,\beta)}(z) + \widetilde{\beta}_n P_n^{(\alpha,\beta)}(z) + \widetilde{\gamma}_n P_{n-1}^{(\alpha,\beta)}(z) - P_{n+1}^{(\alpha-1,\beta-1)}(\xi). \tag{16}$$

Proof. From (6) we have

$$(n+1)P_n^{(\alpha,\beta)}(z) = \frac{d}{dz}((z-\xi)P_n(z)),$$

therefore, by using the forward shift operator (12), we have

$$(z-\xi)P_n(z) = (n+1)\int_{\xi}^{z} P_n^{(\alpha,\beta)}(z)\,dz = \int_{\xi}^{z} \frac{d}{dz}\left(P_{n+1}^{(\alpha-1,\beta-1)}(z)\right)dz. \tag{17}$$

From (17) the identity (14) follows. From (14) expression and using the second structure relation (9) the expression (16) follows. Let us now to prove (15).

By using the second-order differential operator and the forward shift operator of the Jacobi polynomials, we obtain

$$(1-z^2)\frac{d}{dz}P_n^{(\alpha,\beta)}(z) + (\beta-\alpha-z(\alpha+\beta))P_n^{(\alpha,\beta)}(z) = -(\alpha+\beta+n)P_{n+1}^{(\alpha-1,\beta-1)}(z). \tag{18}$$

By using the differential equation (18) and the identity

$$\frac{f(z)g(z)-f(\xi)g(\xi)}{z-\xi} = \frac{f(z)-f(\xi)}{z-\xi}g(z) + f(\xi)\frac{g(z)-g(\xi)}{z-\xi},$$

then (15) follows and hence the result holds. □

The following additional property of orthogonality holds.

Theorem 1. *The polar Jacobi polynomial P_n with pole $\xi \in \mathbb{C}$ fulfills the following property of orthogonality:*

$$\int_{\gamma}\left(P_n(z) + (z-\xi)\frac{d}{dz}P_n(z)\right)P_m^{(\alpha,\beta)}(z)\omega(z;\alpha,\beta)dz = \begin{cases} 0, & m \neq n, \\ (n+1)\left\|P_n^{(\alpha,\beta)}(z)\right\|^2, & m = n. \end{cases} \tag{19}$$

Furthermore, if $n > 1$, then

$$\int_{\gamma}(z-\xi)P_n(z)P_m^{(\alpha,\beta)}(z)\omega(z;\alpha,\beta)dz = \begin{cases} -\dfrac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}P_{n+1}^{(\alpha-1,\beta-1)}(\xi), & m = 0, \\ 0, & 0 < m < n-1, \\ \widetilde{\gamma}_n\left\|P_{n-1}^{(\alpha,\beta)}(z)\right\|^2, & n-1 = m, \\ \widetilde{\beta}_n\left\|P_n^{(\alpha,\beta)}(z)\right\|^2, & n = m, \\ \left\|P_{n+1}^{(\alpha,\beta)}(z)\right\|^2, & n+1 = m, \\ 0, & n+1 < m. \end{cases} \tag{20}$$

Proof. Taking into account (6) we have

$$\int_{\gamma}\left(P_n(z) + (z-\xi)\frac{d}{dz}P_n(z)\right)P_m^{(\alpha,\beta)}(z)\omega(z;\alpha,\beta)\,dz = (n+1)\int_{\gamma} P_n^{(\alpha,\beta)}(z)P_m^{(\alpha,\beta)}(z)\omega(z;\alpha,\beta)\,dz.$$

So, the first property of orthogonality follows. By using the relation (16) and considering the property of orthogonality of the Jacobi polynomials, the second property of orthogonality follows. Hence, the result holds. □

Theorem 2. *The sequence of polar Jacobi polynomials* (P_n) *with pole* $\xi \in \mathbb{C}$ *satisfies the following recurrence relation:*

$$P_{n+1}(z) = zP_n(z) + a_n P_n(z) + b_n P_{n-1}(z) + P_{n+1}^{(\alpha-1,\beta-1)}(\xi), \quad n = 0, 1, ..., \tag{21}$$

with initial conditions $P_{-1}(z) = 0$, $P_0(z) = 1$, *and coefficients*

$$a_n = \frac{(\alpha+\beta-2)(\alpha-\beta)}{(\alpha+\beta+2n)(\alpha+\beta+2n+2)}, \quad b_n = -\frac{4(n+1)(\alpha+n)(\beta+n)(\alpha+\beta+n-1)}{(\alpha+\beta+2n-1)(\alpha+\beta+2n)^2(\alpha+\beta+2n+1)}.$$

Proof. Let the sequence $(\nu_{n,k})$ be such that

$$(z-\xi)P_n(z) = \sum_{k=0}^{n+1} \nu_{n,k} P_k(z).$$

Then by using (6) we obtain

$$\begin{aligned} (z-\xi)\Big(P_n(z) + (n+1)P_n^{(\alpha,\beta)}(z)\Big) &= (z-\xi)\big(P_n(z) + ((z-\xi)P_n(z))'\big) \\ &= \sum_{k=0}^{n+1} \nu_{n,k}\Big(P_k(z) + (z-\xi)\frac{d}{dz}P_k(z)\Big). \end{aligned} \tag{22}$$

By the property of orthogonality (19), we have

$$\sum_{k=0}^{n+1} v_{n,k} \int_\gamma \Big(P_k(z) + (z-\xi)\frac{d}{dz}P_k(z)\Big) P_m^{(\alpha,\beta)}(z)\omega(z;\alpha,\beta)\,dz = \alpha_{n,m}(m+1)\Big\|P_m^{(\alpha,\beta)}(z)\Big\|^2, \tag{23}$$

for $m = 0, 1, ..., n$.

On the other hand, let us denote

$$I_{n,m} = \int_\gamma (z-\xi)P_m^{(\alpha,\beta)}(z)\Big(P_n(z) + (n+1)P_n^{(\alpha,\beta)}(z)\Big)\omega(z;\alpha,\beta)\,dz.$$

From the orthogonality of the Jacobi polynomials and the property of orthogonality (20) we get

$$I_{n,m} = \begin{cases} -P_{n+1}^{(\alpha-1,\beta-1)}(\xi)\dfrac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}, & m = 0, \\ 0, & 0 < m < n-1, \\ -\widetilde{\gamma}_n(\alpha+\beta+n-1)\Big\|P_{n-1}^{(\alpha,\beta)}(z)\Big\|^2, & n-1 = m \\ \Big(\dfrac{2-\alpha-\beta}{2}\widetilde{\beta}_n - \xi(n+1)\Big)\Big\|P_n^{(\alpha,\beta)}(z)\Big\|^2, & n = m. \end{cases} \tag{24}$$

Thus, multiplying (22) by $P_m^{(\alpha,\beta)}(z)$, integrating over γ, and using (23) and (24), we obtain

$$\nu_{n,m} = \frac{I_{n,m}}{(m+1)\Big\|P_m^{(\alpha,\beta)}(z)\Big\|^2} = \begin{cases} -P_{n+1}^{(\alpha-1,\beta-1)}(\xi), & m = 0, \\ 0, & 0 < m < n-1, \\ -b_n, & m = n-1, \\ \dfrac{2-\alpha-\beta}{2(n+1)}\widetilde{\beta}_n - \xi, & m = n. \end{cases}$$

The expression (21) is obtained after a straightforward calculation. □

A direct consequence of this result is the following.

Corollary 1 (The polar ultraspherical case)**.** *The sequence of symmetric polar Jacobi polynomials with pole $\xi \in \mathbb{C}$, i.e., the sequence of polar ultraspherical polynomial with pole $\xi \in \mathbb{C}$ satisfies, namely P_n, the recurrence relation:*

$$\mathsf{P}_{n+1}(z) = z\mathsf{P}_n(z) - \frac{(n+1)(2\alpha+n-1)}{(2\alpha+2n-1)(2\alpha+2n+1)}\mathsf{P}_{n-1}(z) + \mathsf{P}_{n+1}^{(\alpha-1,\alpha-1)}(\xi), \quad n = 0,1,..., \tag{25}$$

with initial conditions $\mathsf{P}_{-1}(z) = 0$, $\mathsf{P}_0(z) = 1$.

Another direct consequence is due the fact when one, or both, of the parameters is a negative integer, then we can factorize the Jacobi polynomial. In fact,

$$P_n^{(-k,\beta)}(z) = (z-1)^k P_{n-k}^{(k,\beta)}, \tag{26}$$
$$P_n^{(\alpha,-k)}(z) = (z+1)^k P_{n-k}^{(\alpha,k)}. \tag{27}$$

Remark 1. *Since in some results we will consider the polar Jacobi polynomials with different parameters, and poles. To avoid such possible confusion, we will denote by $P_n(z;\alpha,\beta;\xi)$ the polar Jacobi polynomial of degree n, parameters α and β, and pole at ξ.*

Corollary 2. *[The factorization] For any positive integer k, the following identities hold:*

$$P_{n+k}(z;-k,\beta;1) = (z-1)^k P_n^{k+1,\beta-1}(z) \tag{28}$$
$$= (z-1)^k \Big((z-1)P_{n-1}(z;k+2,\beta;1) + P_n^{k+1,\beta-1}(1) \Big), \tag{29}$$
$$P_{n+k}(z;\alpha,-k;-1) = (z+1)^k P_n^{\alpha-1,k+1}(z) \tag{30}$$
$$= (z+1)^k \Big((z+1)P_{n-1}(z;\alpha,k+2;-1) + P_n^{\alpha-1,k+1}(-1) \Big). \tag{31}$$

Moreover, the recurrence coefficients satisfy the relations:

$$a_{n+k}(-k,\beta;1) = a_{n-1}(k+2,\beta;1), \quad b_{n+k}(-k,\beta;1) = b_{n-1}(k+2,\beta;1),$$

and

$$a_{n+k}(\alpha,-k;-1) = a_{n-1}(\alpha,k+2;-1), \quad b_{n+k}(\alpha,-k;-1) = b_{n-1}(\alpha,k+2;-1).$$

Proof. The identities (28), (29), (30) and (31) follow by using the factorization of the Jacobi polynomials (26) and (27). In order to obtain the relation between the recurrence coefficients defined in Theorem 2, we must use the former factotization(s) and after a straightforward calculations the identities follow. □

The last result of this section is due the parity relation of the Jacobi polynomials, i.e.

$$P_n^{(\alpha,\beta)}(z) = (-1)^z P_n^{(\beta,\alpha)}(-z). \tag{32}$$

Lemma 2. *For any $\xi \in \mathbb{C}$, the following identity holds:*

$$P_n(z;\alpha,\beta;\xi) = (-1)^n P_n(-z;\beta,\alpha;-\xi). \tag{33}$$

Proof. Starting from 14 and using (32) we have

$$P_n(z;\alpha,\beta;\xi) = \frac{P_{n+1}^{\alpha-1,\beta-1}(z) - P_{n+1}^{\alpha-1,\beta-1}(\xi)}{z-\xi} = (-1)^{n+1}\frac{P_{n+1}^{\beta-1,\alpha-1}(-z) - P_{n+1}^{\beta-1,\alpha-1}(-\xi)}{z-\xi}$$
$$= (-1)^n \frac{P_{n+1}^{\beta-1,\alpha-1}(-z) - P_{n+1}^{\beta-1,\alpha-1}(-\xi)}{-z-(-\xi)} = (-1)^n P_n(-z;\beta,\alpha;-\xi).$$

□

3. Zero location

Finding the roots of polynomials is a problem of interest in both mathematics and in areas of application such as physical systems, which can be reduced to solving certain equation. There are very interesting geometric relationships between the roots of a polynomial $f_n(z)$ and those of $f'_n(z)$. The most important result is the following.

Theorem 3 (The Gauß-Lucas theorem [15]). *Let $f_n(z) \in \mathbb{C}[z]$ be a polynomial of degree at least one. All zeros of $f'_n(z)$ lie in the convex hull of the zeros of the zeros of $f_n(z)$.*

In this section we are going to study the zero distribution for the polar Jacobi polynomials. The next useful result, that was obtained by G. Szegő, let us estimate where such zeros are located.

Theorem 4 (Szegő's theorem [20], [8]). *Let $a(z)$ and $b(z)$ be the polynomials:*

$$a(z) = \sum_{\ell=0}^{n} a_\ell \binom{n}{\ell} z^\ell, \quad b(z) = \sum_{\ell=0}^{n} b_\ell \binom{n}{\ell} z^\ell.$$

If the zeros of $a(z)$ lie in a closed disk $\overline{D}$ and λ_1, ..., λ_n are the zeros of $b(z)$, then the zeros of the "composition" of the two

$$c(z) = \sum_{\ell=0}^{n} a_\ell b_\ell \binom{n}{\ell} z^\ell,$$

has the form $\lambda_\ell \gamma_\ell$, where $\gamma_\ell \in \overline{D}$.

By using this result we are going to locate the circle where all the zeros of the polar Jacobi are located.

Theorem 5. *For any $\Re\alpha, \Re\beta > -1$ and $\xi \in \mathbb{C}$, the zeros of $P_n(z;\alpha,\beta;\xi)$ lie inside the closed disk $\overline{D}(0, 2 + |\xi|)$.*

Proof. Starting from (6) and assuming that

$$a(z) = P_n^{(\alpha,\beta)}(w) = \sum_{k=0}^{n} \mu_k \omega^k, \quad c(z) = P_n(z;\alpha,\beta;w) = \sum_{k=0}^{n} \eta_k \omega^k,$$

where $w := z - \xi$, then $\eta_k = (n+1)/(k+1)\mu_k$. In order to apply Szegő's theorem, we consider

$$b(z) = \sum_{k=0}^{n} \binom{n}{k} \frac{n+1}{k+1} w^k = \sum_{k=0}^{n} \binom{n+1}{k+1} w^k = \frac{(w+1)^{n+1} - 1}{w}.$$

If $b(w_1) = 0$ then $|w_1 + 1| = 1$, so $|z_1| \leq 2 + |\xi|$. Moreover, if $a(z_2) = 0$ then $|z_2| \leq 1$. Therefore, combining these inequalities and applying Szegő's theorem one gets that if $c(z_3) = 0$ then $|z_3| \leq 2 + |\xi|$ and hence the result follows. □

In Figure 1 we illustrate for one hand how accurate the Theorem 5 is, and for the other, we show the behavior of the zeros of the same polar Jacobi polynomial when the pole travels along a specific circle.

In Figure 2 we illustrate an example of Jacobi polar polynomials where the parameters $\Re\alpha \leq -1$ or $\Re\beta \leq -1$, therefore the zeros of the Jacobi polynomial can move away from the interval $[-1, 1]$ in a somewhat uncontrolled way. Therefore Theorem 5 cannot be applied in such a cases. However, observe that in the considered example $-2 < \Re(\alpha + \beta) = -1.95 < -1$.

The next theorem gives the location of the zeros of the polar Jacobi polynomial of degree n and its multiplicity, or equivalently, the location of source points and its corresponding strength.

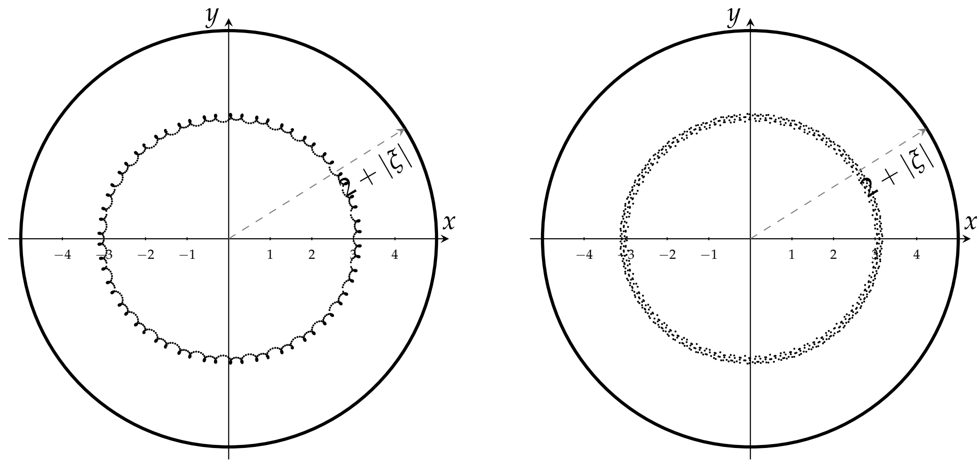

Figure 1. Left: Zeros of the polar Jacobi polynomial $P_{30}(z; 1/2, 2; 3\exp(2\pi kI/30))$ for $k = 0, 1, ..., 29$. Right: Zeros of the polar Jacobi polynomial $P_{30}(z; \sqrt{3}, \pi; 3\exp(2\pi kI/23))$ for $k = 0, 1, ..., 22$.

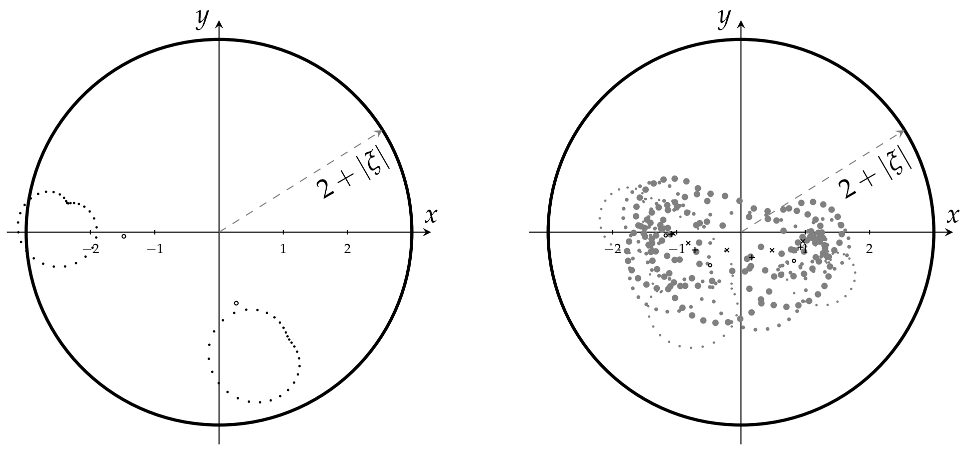

Figure 2. Left: Zeros of the polar Jacobi polynomial $P_2(z; -1/2 + I, -1.45 - I/2; \exp(2\pi kI/30))$ for $k = 0, 1, ..., 29$ (dots) and zeros of the Jacobi polynomials $P_2^{(-1/2+I,-1.45-I/2)}(z)$ (circles). Right: Zeros of the polar Jacobi polynomial $P_n(z; -1/2 + I, -1.45 - I/2; \exp(2\pi kI/30))$ for $k = 0, 1, ..., 29$ (gray dots) and zeros of the Jacobi polynomials $P_n^{(-1/2+I,-1.45-I/2)}(z)$ (+, ×, and circles) for $k = 0, 1, ..., 29$, $n = 3, 4, 5$.

Theorem 6. *For any $\Re\alpha, \Re\beta > -1$ and $\xi \in \mathbb{C}$. The following statements hold:*

1. *If $\zeta \in \mathbb{C}^*$ is a zero of $P_n(z; \alpha, \beta; \xi)$, then $z = -\zeta$ is a zero of $P_n(z; \beta, \alpha; -\xi)$.*
2. *If $\zeta \in \mathbb{C}^*$ is a zero of $P_n^{(\alpha,\beta)}(z)$, then ζ is a zero of P_n.*
3. *The zeros of P_n have multiplicity at most 2 and their multiple zeros are located on $[-1, 1]$.*
4. *All the zeros of P_n are located on the curve*

$$\mathscr{Z}_n(\xi) = \left\{ z \in \mathbb{C} \ : \ P_{n+1}^{(\alpha-1,\beta-1)}(z) = P_{n+1}^{(\alpha-1,\beta-1)}(\xi) \right\} \setminus \{\xi\}. \tag{34}$$

Proof. The first statement holds true due (34), the second statement holds true due (6), and the forth statement holds true due (14).

Assume ω is a zero P_n of multiplicity greater than two then, by (6), ω is a zero of $P_n^{(\alpha,\beta)}$ and also a zero of $\left(P_n^{(\alpha,\beta)}\right)'$; thus ω is a zero of multiplicity 2 of $P_n^{(\alpha,\beta)}$. This is a contradiction since the zeros of the Jacobi polynomials are all simple. Therefore, statements 3 holds true. □

Remark 2.
- *Observe that the zeros of P_n do not have to be simple. Let $\xi_+ = (1+2\sqrt{6})/5$ or $\xi_- = (1-2\sqrt{6})/5$, then the polar polynomial of degree two $P_2(z;0,1,\xi_+) = \left(z - \frac{1-\sqrt{6}}{5}\right)^2$, or $P_2(z;0,1,\xi_-) = \left(z - \frac{1+\sqrt{6}}{5}\right)^2$.*
- *When the parameters are nor standard, i.e., $\Re\alpha < -1$ or $\Re\beta < -1$ then, by Corollary 2 then statement 3 of Theorem 6 is no longer true. For example if $\alpha = -4$, $\beta = 1 > 0$, and $n = 5$, then $P_5(z;-4,1,1) = (z-1)^4(z-5/7)$.*

We can establish the following result concerning the boundedness of the zeros of the polar polynomials.

Lemma 3. *Given $\xi \in \mathbb{C}$ let us define the two numbers $\Delta_\xi := \sup\{|\xi - z| : z \in [-1,1]\}$, and $\delta_\xi := \inf\{|\xi - z| : z \in [-1,1]\}$. Then*
1. *All zeros of the polar Jacobi polynomials with pole ξ are contained in $|z| \leq \Delta_\xi + 1$.*
2. *If $\delta_\xi > 1$, the zeros of the polar Jacobi polynomials with pole ξ are simple and contained in the exterior of the ellipse $|z+1| + |z-1| = 2\alpha$, where $1 < \alpha < \delta_\xi$.*

Proof. By (14) the zeros of $P_n(z)$ are located in $\mathscr{Z}_n(\xi)$. Since $\left|P_{n+1}^{(\alpha-1,\beta-1)}(\xi)\right| < \Delta_\xi^{n+1}$, they are contained in the interior of the set $\left|P_{n+1}^{(\alpha-1,\beta-1)}(z)\right| = \Delta_\xi^{n+1}$. It is known the zeros of $P_{n+1}^{(\alpha-1,\beta-1)}(z)$, namely $x_{n+1,k}$, satisfy $|x_{n+1,k}| \leq 1$. Therefore, for any $t \in \mathbb{C}$, such that $|t| > 1 + \Delta_\xi$, we have

$$\left|P_{n+1}^{(\alpha-1,\beta-1)}(z)\right| = \prod_{k=0}^{n} |z - x_{n+1,k}| \geq \prod_{k=0}^{n} \big||z| - |x_{n+1,k}|\big| > \Delta_\xi^{n+1},$$

hence the first statement holds.

About the second statement, let z be such that $|z+1| + |z-1| = 2\alpha$. From the well-known arithmetic-geometric mean inequality we have

$$\left|P_{n+1}^{(\alpha-1,\beta-1)}(z)\right| \leq \left(\frac{1}{n+1}\sum_{k=0}^{n} |z - x_{n+1,k}|\right)^{n+1} < \alpha^{n+1}.$$

If ω is a zero of P_n, from (34) we get

$$\left|P_{n+1}^{(\alpha-1,\beta-1)}(\omega)\right| = \left|P_{n+1}^{(\alpha-1,\beta-1)}(\xi)\right| = \prod_{k=0}^{n} |\xi - x_{n+1,k}| > \delta_\xi^{n+1} > \alpha^{n+1}.$$

Therefore, the result holds. □

The last result is about the asymptotic behavior of the zeros of the polar Jacobi polynomials.

Theorem 7 (Theorem 22 in [7]). *The accumulation points of zeros of (P_n) are located on the set $\mathscr{Z}(\xi) \cup [-1,1]$, where $\mathscr{Z}(\xi)$ is the ellipse*

$$\mathscr{Z}(\xi) = \{z \in \mathbb{C} : z = \cosh(\log|\varphi(\xi)| + i\theta),\ 0 \leq \theta < 2\pi\} = \left\{z \in \mathbb{C} : \left|z + \sqrt{z^2-1}\right| = |\varphi(\xi)|\right\},$$

where $\varphi(z) = z + \sqrt{z^2 - 1}$.

4. Concluding remarks

The Jacobi polynomials are part of the scheme of orthogonal hypergeometric functions, and since practically all the elements used in this work are known within this scheme, therefore, the work could be extended to the classical polynomials without much difficulty. It would be natural to consider doing analogous work for discrete classical polynomials by replacing the derivative operator by the backward (or forward) difference operator.

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