Double summation addition theorems for Jacobi functions of the first and second kind

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Abstract. In this paper, we review and derive hyperbolic and trigonometric double summation addition theorems for Jacobi functions of the first and second kind. In connection with these addition theorems, we perform a full analysis of the relation between symmetric, antisymmetric, and odd-half-integer parameter values for the Jacobi functions with certain Gauss hypergeometric functions that satisfy a quadratic transformation, including associated Legendre, Gegenbauer and Ferrers functions of the first and second kind. We also introduce Olver normalizations of the Jacobi functions, which are particularly useful in the derivation of expansion formulas when the parameters are integers. We introduce an application of the addition theorems for the Jacobi functions of the second kind to separated eigenfunction expansions of a fundamental solution of the Laplace-Beltrami operator on the compact and noncompact rank one symmetric spaces.

Key words: Addition theorems; Jacobi function of the first kind; Jacobi function of the second kind; Jacobi polynomials; ultraspherical polynomials

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Dedicated to Dick Askey whose favorite function was the Jacobi polynomial.

1 Introduction

Jacobi polynomials (hypergeometric polynomials) were introduced by the German mathematician Carl Gustav Jacob Jacobi (1804–1851). These polynomials first appear in an article by Jacobi, which was published posthumously in 1859 by Heinrich Eduard Heine [28]. Jacobi polynomials, $P_n^{(\alpha,\beta)}(x)$, are polynomials which

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for $\Re\alpha, \Re\beta > -1$ are orthogonal on the real segment $[-1,1]$ [29, (9.8.2)] and can be defined in terms of a terminating sum as follows:

$$P_n^{(\alpha,\beta)}(\cos\theta) := \frac{\Gamma(\alpha+1+n)}{\Gamma(n+1)\,\Gamma(\alpha+\beta+1+n)} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{\Gamma(\alpha+\beta+1+n+k)}{\Gamma(\alpha+1+k)} \sin^{2k}(\tfrac{1}{2}\theta), \tag{1}$$

where Γ is the gamma function [48, (5.2.1)], and $\binom{n}{k}$ the binomial coefficient [48, (1.2.1)]. The above definition of the Jacobi polynomial is equivalent to the following Gauss hypergeometric representation [48, (18.5.7)]:

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!} {}_2F_1\left(\begin{matrix} -n, n+\alpha+\beta+1 \\ \alpha+1 \end{matrix}; \frac{1-x}{2}\right), \tag{2}$$

where $x = \cos\theta$. We will return to the notations used in (2) in the following section.

Ultraspherical polynomials, traditionally defined by [48, (18.7.1)]

$$C_n^\lambda(\cos\theta) := \frac{\Gamma(\lambda+\frac{1}{2})\Gamma(2\lambda+n)}{\Gamma(2\lambda)\Gamma(\lambda+\frac{1}{2}+n)} P_n^{(\lambda-\frac{1}{2},\lambda-\frac{1}{2})}(\cos\theta), \tag{3}$$

are symmetric $\alpha=\beta$ Jacobi polynomials. These polynomials are commonly referred to as Gegenbauer polynomials after Austrian mathematician Leopold Gegenbauer (1849–1903). However the Czech (Austrian) astronomer and mathematician Moriz Allé discovered and used many of their fundamental properties including their generating function *and* addition theorem [2] almost a decade prior to Gegenbauer [19, 20], and Heine [24, p. 455]. See the nice discussion of the history of the addition theorem for ultraspherical polynomials by Koornwinder in [38, p. 383]. The addition theorem for ultraspherical polynomials is given by [48, (18.18.8)]

$$\begin{aligned} &C_n^\lambda(\cos\theta_1\cos\theta_2 \pm \sin\theta_1\sin\theta_2\cos\phi) \\ &\quad = \frac{n!}{(2\lambda)_n} \sum_{k=0}^{n} \frac{(\mp 1)^k (\lambda)_k (2\lambda)_{2k}}{(-n)_k (\lambda-\frac{1}{2})_k (2\lambda+n)_k} (\sin\theta_1\sin\theta_2)^k C_{n-k}^{\lambda+k}(\cos\theta_1) C_{n-k}^{\lambda+k}(\cos\theta_2) C_k^{\lambda-\frac{1}{2}}(\cos\phi). \end{aligned} \tag{4}$$

This result is quite important all by itself. In the special case $\lambda = \frac{1}{2}$, it becomes one way of writing the addition theorem for spherical harmonics on the two-dimensional sphere:

$$P_n(\cos\theta_1\cos\theta_2 \pm \sin\theta_1\sin\theta_2\cos\phi) = \sum_{k=-n}^{n} (\pm 1)^k \frac{(n-k)!}{(n+k)!} \mathsf{P}_n^k(\cos\theta_1) \mathsf{P}_n^k(\cos\theta_2) \cos(k\phi), \tag{5}$$

where the P_n^k are Ferrers functions of the first kind [48, (14.3.1)]. See the foreword of Willard Miller (1977) [44] written by Richard Askey for a beautiful discussion (on pp. xix–xx) of addition theorems (see also [1, §2.7]).

Given the addition theorem for ultraspherical polynomials (4), it was a natural problem to extend this to Jacobi polynomials for $\alpha \neq \beta$. It was a good match when Richard Askey, on sabbatical at the Mathematical Centre in Amsterdam during 1969–1970, met Tom Koornwinder there, who had some experience with group theoretical methods and was looking for a good subject for a Ph.D. thesis. Askey suggested to Koornwinder the problem of finding an addition theorem for Jacobi polynomials, and he also arranged that Koornwinder could attend a special year at the Mittag-Leffler Institute in Sweden. There Koornwinder obtained the desired result [35, 34, 36, 37]. He later found that his group theoretic method and the resulting addition theorem in a special case were anticipated by two papers in Russian: Vilenkin and Šapiro [51] realized that disk polynomials [48, (18.37.1)] and in particular the Jacobi polynomials $P_n^{(\alpha,0)}$, α an integer, can be interpreted as spherical functions on the complex projective space $\mathrm{SU}(\alpha+2)/\mathrm{U}(\alpha+1)$ [6, 7] or as spherical functions on the complex unit sphere $\mathrm{U}(\alpha+2)/\mathrm{U}(\alpha+1)$ in $\mathbb{C}^{\alpha+2}$ as a homogeneous space of the unitary group $\mathrm{U}(\alpha+2)$ (see references by Ikeda, Kayama and Seto in [34, 35]). Šapiro obtained from that observation, the addition theorem for Jacobi polynomials in the $\beta = 0$ case [52].

Koornwinder initially presented his addition theorem for Jacobi polynomials in a series of three papers in 1972 [34, 36, 37]. Koornwinder gave four different proofs of the addition formula for Jacobi polynomials. His first proof focused on the spherical functions of the Lie group $\mathrm{U}(d)/\mathrm{U}(d-1)$, $d \geq 2$ integer, and appeared in [36, 37]; his second proof that used ordinary spherical harmonics appeared in [30]; his third proof was an analytic proof and it appeared in [4, 31, 32]; and a short proof using orthogonal polynomials in three variables which appeared in [33].

Let us consider the trigonometric context of Koornwinder's addition theorem for the Jacobi polynomials. Let $n \in \mathbb{N}_0$, $\alpha > \beta > -\frac{1}{2}$, $\cos\theta_1 = \frac{1}{2}(\mathrm{e}^{i\theta_1} + \mathrm{e}^{-i\theta_1})$, $\cos\theta_2 = \frac{1}{2}(\mathrm{e}^{i\theta_2} + \mathrm{e}^{-i\theta_2})$, $w \in (-1,1)$, $\phi \in [0,\pi]$. Then Koornwinder's addition theorem for Jacobi polynomials is given by

$$P_n^{(\alpha,\beta)}\left(2|\cos\theta_1\cos\theta_2 \pm \mathrm{e}^{i\phi} w \sin\theta_1 \sin\theta_1|^2 - 1\right) = \frac{n!\Gamma(\alpha+1)}{\Gamma(\alpha+n+1)} \sum_{k=0}^{n} \frac{(\alpha+1)_k(\alpha+\beta+n+1)_k}{(\alpha+k)(\beta+1)_k(-n)_k}$$

$$\times \sum_{l=0}^{k} (\mp 1)^{k-l} \frac{(\alpha+k+l)(-\beta-n)_l}{(\alpha+n+1)_l} (\cos\theta_1\cos\theta_2)^{k-l} (\sin\theta_1\sin\theta_2)^{k+l}$$

$$\times P_{n-k}^{(\alpha+k+l,\beta+k-l)}(\cos(2\theta_1))\, P_{n-k}^{(\alpha+k+l,\beta+k-l)}(\cos(2\theta_2))\, w^{k-l} P_l^{(\alpha-\beta-1,\beta+k-l)}(2w^2-1) \frac{\beta+k-l}{\beta} C_{k-l}^{\beta}(\cos\phi). \quad (6)$$

As we will see in Section 3 below, this addition theorem and its various counterparts for Jacobi functions of the first and second kind are deeply connected to a 2-variable orthogonal polynomial system sometimes referred to as parabolic biangle polynomials $\mathcal{P}_{k,l}^{(\alpha,\beta)}(w,\phi)$.

In the case of ultraspherical and Jacobi polynomials, the sum is terminating, as one would expect since the object of study is a polynomial. However, as Flensted-Jensen and Koornwinder realized [17], the addition theorem for Jacobi polynomials can be extended to Jacobi functions of the first kind by formally taking the outer sum limit to infinity. While Jacobi polynomials $P_n^{(\alpha,\beta)}$ have n discrete, the Jacobi functions $\varphi_\lambda^{(\alpha,\beta)}$ have λ continuous (see (109) below). When one starts to consider Jacobi functions, then many new questions arise which must be understood for a full theory of the separated eigenfunctions expansions of Jacobi functions. First of all, one must consider two separate contexts, the trigonometric context where the arguments of the functions are analytically continued from the segment $(-1,1)$ and also the hyperbolic context, where the arguments of the functions are analytically continued from the segment $(1,\infty)$. On top of that, one must also consider the particular expansions of Jacobi functions of the second kind. Gegenbauer and Jacobi functions are solutions to second-order ordinary differential equations. Therefore there are two linearly independent solutions, namely the functions of the first kind and the functions of the second kind. The separated eigenfunction expansions of the Gegenbauer functions of the first and second kind were treated quite extensively in a paper by Durand, Fishbane and Simmons (1976) [14]. Durand extended Koornwinder's addition theorem to Jacobi (and other) functions of the second kind in [13].

The study of multi-summation addition theorems for Jacobi functions of the first and second kind seems not to have moved forward since the advances by Durand and by Flensted-Jensen and Koornwinder. In the remainder of this paper, we give the full multi-summation expansions of Jacobi functions of the second kind and bring the full theory of the expansions of Jacobi functions to a circle.

Addition theorems, such as the addition theorem for Jacobi polynomials, is intimately related to separated eigenfunction expansions of spherical functions (reproducing kernels) on Riemannian symmetric spaces. For instance, the special argument of Gegenbauer's addition theorem (4) is easily expressible in terms of the geodesic distance between two arbitrary points on the d-dimensional real hypersphere. One is often interested in eigenfunction expansions of a fundamental solution of Laplace's equation because this allows you to perform a multipole expansion of arbitrarily shaped mass distributions. When you perform a global analysis of the Laplacian (Laplace-Beltrami operator) on rank one symmetric spaces, a fundamental solution of the Laplacian is given in terms of radial solutions of a second order differential equation and since the solutions are singular at the origin, the solutions are given in terms of the functions of the second kind. In the particular case of rank one symmetric spaces beyond the real case (complex, quaternionic and octonionic), then these fundamental solutions are given in terms of Jacobi functions of the second kind. This is the motivation of the work for the present paper. We were originally motivated by the symmetric spaces. They show up in the solution to the problem we were trying to solve and the project evolved from the singular part which arises within the function of the second kind.

2 Preliminaries

Throughout this paper we adopt the following set notations: $\mathbb{N}_0 := \{0\} \cup \mathbb{N} = \{0, 1, 2, 3, \ldots\}$, and we use the set $\mathbb{C}$ which represents the complex numbers. Jacobi functions (and their special cases such as Gegenbauer, associated Legendre and Ferrers functions) have representations given in terms of Gauss hypergeometric functions which can be defined in terms of an infinite series over ratios of shifted factorials (Pochhammer symbols). The shifted factorial can be defined for $a \in \mathbb{C}$, $n \in \mathbb{N}_0$ by [48, (5.2.4), (5.2.5)] $(a)_n := (a)(a+1)\cdots(a+n-1)$. The following ratio of two gamma functions [48, Chapter 5] are related to the shifted factorial, namely for $a \in \mathbb{C} \setminus -\mathbb{N}_0$, one has

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}, \tag{7}$$

which allows one to extend the definition to non-positive integer values of n. Some other properties of shifted factorial which we will use are $(n, k \in \mathbb{N}_0,\ n \geq k)$

$$\Gamma(a-n) = \frac{(-1)^n \Gamma(a)}{(1-a)_n}, \tag{8}$$

$$(-n)_k = \frac{(-1)^k n!}{(n-k)!}. \tag{9}$$

One also has the following expression for the generalized binomial coefficient for $z \in \mathbb{C}$, $n \in \mathbb{N}_0$ [48, (1.2.6)]

$$\binom{z}{n} = \frac{(-1)^n (-z)_n}{n!}. \tag{10}$$

Define the multisets $\mathbf{a} := \{a_1, \ldots, a_r\}$, $\mathbf{b} := \{b_1, \ldots, b_s\}$. We will also use the common notational product convention, $a_l \in \mathbb{C}$, $l \in \mathbb{N}$, $r \in \mathbb{N}_0$, e.g.,

$$(\mathbf{a})_k := (a_1, \ldots, a_r)_k := (a_1)_k (a_2)_k \cdots (a_r)_k, \tag{11}$$

$$\Gamma(\mathbf{a}) := \Gamma(a_1, \ldots, a_r) := \Gamma(a_1) \cdots \Gamma(a_r). \tag{12}$$

Also define the multiset notation $\mathbf{a} + t := \{a_1 + t, \ldots, a_r + t\}$.

For any expression of the form $(z^2-1)^\alpha$, we fix the branch of the power functions such that

$$(z^2-1)^\alpha := (z+1)^\alpha (z-1)^\alpha,$$

for any fixed $\alpha \in \mathbb{C}$ and $z \in \mathbb{C} \setminus \{-1, 1\}$. The generalized hypergeometric function [48, Chapter 16] is defined by the infinite series [48, (16.2.1)]

$${}_rF_s(\mathbf{a}; \mathbf{b}; z) := {}_rF_s\left(\begin{matrix}\mathbf{a}\\ \mathbf{b}\end{matrix}; z\right) := \sum_{k=0}^\infty \frac{(\mathbf{a})_k}{(\mathbf{b})_k} \frac{z^k}{k!}, \tag{13}$$

where $|z| < 1$, $b_j \notin -\mathbb{N}_0$, for $j \in \{1, \ldots, s\}$; and elsewhere by analytic continuation. Further define the Olver normalized (scaled or regularized) generalized hypergeometric series ${}_r\boldsymbol{F}_s(\mathbf{a}; \mathbf{b}; z)$, given by

$${}_r\boldsymbol{F}_s(\mathbf{a}; \mathbf{b}; z) := {}_r\boldsymbol{F}_s\left(\begin{matrix}\mathbf{a}\\ \mathbf{b}\end{matrix}; z\right) := \frac{1}{\Gamma(\mathbf{b})} {}_rF_s\left(\begin{matrix}\mathbf{a}\\ \mathbf{b}\end{matrix}; z\right) = \sum_{k=0}^\infty \frac{(a_1, \ldots, a_r)_k}{\Gamma(\mathbf{b}+k)} \frac{z^k}{k!}, \tag{14}$$

which is entire for all $a_l, b_j \in \mathbb{C}$, $l \in \{1,\ldots,r\}$, $j \in \{1,\ldots,s\}$. Both the generalized and Olver normalized generalized hypergeometric series, if nonterminating, are entire if $r \le s$, convergent for $|z|<1$ if $r=s+1$ and divergent if $r \ge s+1$. The special case of the generalized hypergeometric function with $r=2$, $s=1$ is referred to as the Gauss hypergeometric function [48, Chapter 15], or simply the hypergeometric function. It has many interesting properties, including linear transformations which were discovered by Euler and Pfaff. Euler's linear transformation is [48, (15.8.1)]

$$ {}_2\boldsymbol{F}_1\left(\begin{matrix} a,b \\ c \end{matrix};z\right) = (1-z)^{c-a-b}\,{}_2\boldsymbol{F}_1\left(\begin{matrix} c-a,c-b \\ c \end{matrix};z\right) \tag{15} $$

and Pfaff's linear transformation is [48, (15.8.1)]

$$ {}_2\boldsymbol{F}_1\left(\begin{matrix} a,b \\ c \end{matrix};z\right) = (1-z)^{-a}\,{}_2\boldsymbol{F}_1\left(\begin{matrix} a,c-b \\ c \end{matrix};\frac{z}{z-1}\right) = (1-z)^{-b}\,{}_2\boldsymbol{F}_1\left(\begin{matrix} b,c-a \\ c \end{matrix};\frac{z}{z-1}\right). \tag{16} $$

2.1 The Gegenbauer and associated Legendre functions

The functions which satisfy quadratic transformations of the Gauss hypergeometric function are given by Gegenbauer and associated Legendre functions of the first and second kind. As we will see, these functions correspond to Jacobi functions of the first and second kind when their parameters satisfy certain relations. We now describe some of the properties of these functions, which have a deep and long history.

Let $n \in \mathbb{N}_0$. The Gegenbauer (ultraspherical) polynomial which is an important specialization of the Jacobi polynomial for symmetric parameters values, is given in terms of a terminating Gauss hypergeometric series [48, (18.7.1)]

$$ C_n^\mu(z) = \frac{(2\mu)_n}{(\mu+\frac12)_n} P_n^{(\mu-\frac12,\mu-\frac12)}(z) = \frac{(2\mu)_n}{n!}\,{}_2F_1\left(\begin{matrix} -n, 2\mu+n \\ \mu+\frac12 \end{matrix};\frac{1-z}{2}\right). \tag{17} $$

Note that the ultraspherical polynomials satisfy the following parity relation [48, Table 18.6.1]

$$ C_n^\mu(-z) = (-1)^n C_n^\mu(z). \tag{18} $$

Gegenbauer functions which generalize ultraspherical polynomials with arbitrary degrees $n=\lambda\in\mathbb{C}$ are solutions $w=w(z)=w_\lambda^\mu(z)$ to the Gegenbauer differential equation [48, Table 18.8.1]

$$ (z^2-1)\frac{\mathrm{d}^2 w(z)}{\mathrm{d}z^2} + (2\lambda+1)\,z\frac{\mathrm{d}w(z)}{\mathrm{d}z} - \lambda(\lambda+2\mu)w(z) = 0. \tag{19} $$

There are two linearly independent solutions to this second order ordinary differential equation which are referred to as Gegenbauer functions of the first and second kind $C_\lambda^\mu(z)$, $D_\lambda^\mu(z)$. A closely connected differential equation to the Gegenbauer differential equation (19) is the associated Legendre differential equation which is given by [48, (14.2.1)]

$$ (1-z^2)\frac{\mathrm{d}^2 w(z)}{\mathrm{d}z^2} - 2z\frac{\mathrm{d}w(z)}{\mathrm{d}z} + \left(\nu(\nu+1) - \frac{\mu^2}{1-z^2}\right)w(z) = 0. \tag{20} $$

Two linearly independent solutions to this equation are referred to as associated Legendre functions of the first and second kind $P_\nu^\mu(z)$, $Q_\nu^\mu(z)$. In the following subsection we will present the definitions of these important functions which are Gauss hypergeometric functions which satisfy a quadratic transformation.

2.1.1 Hypergeometric representations of the Gegenbauer and associated Legendre functions

The Gegenbauer function of the first kind is defined by [48, (15.9.15)]

$$C_\lambda^\mu(z) := \frac{\sqrt{\pi}\,\Gamma(\lambda+2\mu)}{2^{2\mu-1}\Gamma(\mu)\Gamma(\lambda+1)}\,{}_2\boldsymbol{F}_1\left(\begin{matrix}-\lambda, 2\mu+\lambda\\ \mu+\frac12\end{matrix};\frac{1-z}{2}\right), \tag{21}$$

where $\lambda+2\mu\not\in-\mathbb{N}_0$. It is a clear extension of the Gegenbauer polynomial when the index is allowed to be a complex number as well as a non-negative integer. Two representations which will be useful for us in comparing to the Jacobi function of the second kind are referred to as Gegenbauer functions of the second kind which have hypergeometric representations given with $\lambda+2\mu\not\in-\mathbb{N}_0$, [14, (2.3)]

$$D_\lambda^\mu(z) := \frac{\mathrm{e}^{i\pi\mu}\Gamma(\lambda+2\mu)}{\Gamma(\mu)(2z)^{\lambda+2\mu}}\,{}_2\boldsymbol{F}_1\left(\begin{matrix}\frac12\lambda+\mu, \frac12\lambda+\mu+\frac12\\ \lambda+\mu+1\end{matrix};\frac{1}{z^2}\right) \tag{22}$$

$$= \frac{\mathrm{e}^{i\pi\mu}2^\lambda\Gamma(\lambda+\mu+\frac12)\Gamma(\lambda+2\mu)}{\sqrt{\pi}\,\Gamma(\mu)(z-1)^{\lambda+\mu+\frac12}(z+1)^{\mu-\frac12}}\,{}_2\boldsymbol{F}_1\left(\begin{matrix}\lambda+1, \lambda+\mu+\frac12\\ 2\lambda+2\mu+1\end{matrix};\frac{2}{1-z}\right), \tag{23}$$

and in the second representation $\lambda+\mu+\frac12\not\in-\mathbb{N}_0$. The equality of these two representations of the Gegenbauer function of the second kind follow from a quadratic transformation of the Gauss hypergeometric function from Group 3 to Group 1 in [48, Table 15.8.1]. The associated Legendre function of the first kind is defined as [48, (14.3.6) and §14.21(i)]

$$P_\nu^\mu(z) := \left(\frac{z+1}{z-1}\right)^{\frac12\mu}\,{}_2\boldsymbol{F}_1\left(\begin{matrix}-\nu, \nu+1\\ 1-\mu\end{matrix};\frac{1-z}{2}\right), \tag{24}$$

where $|1-z|<2$, and elsewhere in z by analytic continuation. The associated Legendre function of the second kind $Q_\nu^\mu:\mathbb{C}\setminus(-\infty,1]\to\mathbb{C}$, $\nu+\mu\not\in-\mathbb{N}$, has the following two single Gauss hypergeometric function representations [48, (14.3.7) and §14.21], [43, entry 24, p. 161],

$$Q_\nu^\mu(z) := \frac{\sqrt{\pi}\,\mathrm{e}^{i\pi\mu}\Gamma(\nu+\mu+1)(z^2-1)^{\frac12\mu}}{2^{\nu+1}z^{\nu+\mu+1}}\,{}_2\boldsymbol{F}_1\left(\begin{matrix}\frac{\nu+\mu+1}{2}, \frac{\nu+\mu+2}{2}\\ \nu+\frac32\end{matrix};\frac{1}{z^2}\right) \tag{25}$$

$$= \frac{2^\nu\mathrm{e}^{i\pi\mu}\Gamma(\nu+1)\Gamma(\nu+\mu+1)(z+1)^{\frac12\mu}}{(z-1)^{\frac12\mu+\nu+1}}\,{}_2\boldsymbol{F}_1\left(\begin{matrix}\nu+1, \nu+\mu+1\\ 2\nu+2\end{matrix};\frac{2}{1-z}\right), \tag{26}$$

and for the second representation, $\nu\not\in-\mathbb{N}$. The first and second single Gauss hypergeometric representations are convergent as a Gauss hypergeometric series for $|z|>1$, respectively $|z-1|>2$, and elsewhere in $z\in\mathbb{C}\setminus(-\infty,1]$ by analytic continuation of the Gauss hypergeometric function.

Remark 1. *The relations between the associated Legendre functions of the first and second kind are related to the Gegenbauer functions of the first and second kind by [48, (14.3.22)]*

$$P_\nu^\mu(z) = \frac{\Gamma(\frac12-\mu)\Gamma(\nu+\mu+1)}{2^\mu\sqrt{\pi}\,\Gamma(\nu-\mu+1)(z^2-1)^{\frac12\mu}}C_{\nu+\mu}^{\frac12-\mu}(z), \tag{27}$$

$$Q_\nu^\mu(z) = \frac{\mathrm{e}^{2\pi i(\mu-\frac14)}\sqrt{\pi}\,\Gamma(\frac12-\mu)\Gamma(\nu+\mu+1)}{2^\mu\Gamma(\nu-\mu+1)(z^2-1)^{\frac12\mu}}D_{\nu+\mu}^{\frac12-\mu}(z), \tag{28}$$

which are valid for $\mu\in\mathbb{C}\setminus\{\frac12,\frac32,\ldots\}$, $\nu+\mu\in\mathbb{C}\setminus-\mathbb{N}$. *Equivalently, the inverse relationships are given by*

$$C_\lambda^\mu(z) = \frac{\sqrt{\pi}\,\Gamma(\lambda+2\mu)}{2^{\mu-\frac12}\Gamma(\mu)\Gamma(\lambda+1)(z^2-1)^{\frac{\mu}{2}-\frac14}}P_{\lambda+\mu-\frac12}^{\frac12-\mu}(z), \tag{29}$$

$$D_\lambda^\mu(z) = \frac{\mathrm{e}^{2\pi i(\mu-\frac14)}\Gamma(\lambda+2\mu)}{\sqrt{\pi}\,2^{\mu-\frac12}\Gamma(\mu)\Gamma(\lambda+1)(z^2-1)^{\frac12\mu-\frac14}}Q_{\lambda+\mu-\frac12}^{\frac12-\mu}(z), \tag{30}$$

which are valid for all $\lambda+2\mu\in\mathbb{C}\setminus-\mathbb{N}_0$.

Remark 2. *By comparing Gauss hypergeometric representations of the various functions, one may express* ${}_2\boldsymbol{F}_1(a, a+\frac12; c; z)$ *in terms of associated Legendre functions of the first and second kind* P_ν^μ, Q_ν^μ *and the Gegenbauer functions of the first and second kind* C_ν^μ, D_ν^μ *using the following very useful formulas. Let* $z \in \mathbb{C} \setminus [1,\infty)$. *Then*

$$ {}_2\boldsymbol{F}_1\left(\begin{matrix} a, a+\frac12 \\ c \end{matrix}; z\right) = 2^{c-1} z^{\frac12(1-c)} (1-z)^{\frac12 c - a - \frac12} P_{2a-c}^{1-c}\left(\frac{1}{\sqrt{1-z}}\right) \tag{31} $$

$$ = \frac{2^{2c-2}\Gamma(c-\frac12)\Gamma(2(a-c+1))}{\sqrt{\pi}\,\Gamma(2a)(1-z)^a} C_{2a-2c+1}^{c-\frac12}\left(\frac{1}{\sqrt{1-z}}\right), \tag{32} $$

where $2c \notin \{1,-1,-3,\ldots\}$, $2a-2c \notin \{-2,-3,\ldots\}$, *and*

$$ {}_2\boldsymbol{F}_1\left(\begin{matrix} a, a+\frac12 \\ c \end{matrix}; z\right) = \frac{\mathrm{e}^{i\pi(c-2a-\frac12)} 2^{c-\frac12} (1-z)^{\frac12 c - a - \frac14}}{\sqrt{\pi}\,\Gamma(2a) z^{\frac12 c - \frac14}} Q_{c-\frac32}^{2a-c+\frac12}\left(\frac{1}{\sqrt{z}}\right) \tag{33} $$

$$ = \frac{\mathrm{e}^{i\pi(2a-c)} 2^{2c-2a-1}\Gamma(c-2a)(1-z)^{c-2a-\frac12}}{\Gamma(2c-2a-1) z^{c-a-\frac12}} D_{2a-1}^{c-2a}\left(\frac{1}{\sqrt{z}}\right), \tag{34} $$

where $c, c-2a \notin -\mathbb{N}_0$.

2.1.2 The Gegenbauer functions on-the-cut $(-1,1)$ and the Ferrers Functions

We will consider Jacobi functions of the second kind on-the-cut in Section 2.2.3. As we will see, for certain combinations of the parameters which we will describe below, the Jacobi functions of the first and second kind on the cut are related to the the Gegenbauer functions of the first and second kind on-the-cut and the associated Legendre functions of the first and second kind on-the-cut (Ferrers functions).

The Gegenbauer functions of the first and second kind on-the-cut are defined in terms of the Gegenbauer functions immediately above and below the segment $(-1,1)$ in the complex plane. These definition are given by [12, (3.3), (3.4)]

$$ \mathsf{C}_\lambda^\mu(x) := D_\lambda^\mu(x+i0) + \mathrm{e}^{-2\pi i\mu} D_\lambda^\mu(x-i0) = C_\lambda^\mu(x \pm i0), \quad x \in (-1,1], \tag{35} $$

$$ \mathsf{D}_\lambda^\mu(x) := -iD_\lambda^\mu(x+i0) + i\mathrm{e}^{-2\pi i\mu} D_\lambda^\mu(x-i0), \quad x \in (-1,1). \tag{36} $$

Note that $\mathsf{C}_\lambda^\mu(x)$ and D_λ^μ are real for real values of λ and μ.

The Ferrers functions of the first and second kind are defined as [48, (14.23.1), (14.23.2)]

$$ \mathsf{P}_\nu^\mu(x) := \mathrm{e}^{\pm i\pi\mu} P_\nu^\mu(x \pm i0) = \frac{i\mathrm{e}^{-i\pi\mu}}{\pi}\left(\mathrm{e}^{-\frac12 i\pi\mu} Q_\nu^\mu(x+i0) - \mathrm{e}^{\frac12 i\pi\mu} Q_\nu^\mu(x-i0)\right), \tag{37} $$

$$ \mathsf{Q}_\nu^\mu(x) := \frac{\mathrm{e}^{-i\pi\mu}}{2}\left(\mathrm{e}^{-\frac12 i\pi\mu} Q_\nu^\mu(x+i0) + \mathrm{e}^{\frac12 i\pi\mu} Q_\nu^\mu(x-i0)\right). \tag{38} $$

Using the above definition one can readily obtain a single hypergeometric representation of the Gegenbauer function of the first kind on-the-cut, namely

$$ \mathsf{C}_\lambda^\mu(x) = \frac{\sqrt{\pi}\,\Gamma(2\mu+\lambda)}{2^{2\mu-1}\Gamma(\mu)\Gamma(\lambda+1)}\, {}_2\boldsymbol{F}_1\left(\begin{matrix} -\lambda, 2\mu+\lambda \\ \mu+\frac12 \end{matrix}; \frac{1-x}{2}\right), \tag{39} $$

which is identical to the Gegenbauer function of the first kind (21) because this function analytically continues to the segment $(-1,1)$, see (35). For the Gegenbauer function of the second kind on-the-cut, one can readily obtain a double hypergeometric representation of by using the definition (36) and then using the interrelation between the Gegenbauer function of the second kind and the Legendre function of the second kind and then comparing to the Ferrers function of the second kind through its definition (38).

However, first we will give hypergeometric representations of the Ferrers function of the first and second kind which are easily found in the literature. The first author recently co-authored a paper with Park and Volkmer where all double hypergeometric representations of the Ferrers function of the second kind were computed [11]. Using (37) one can derive hypergeometric representations of the Ferrers function of the first kind (associated Legendre function of the first kind on-the-cut) $\mathsf{P}_\nu^\mu : (-1,1) \to \mathbb{C}$. For instance, one has a single hypergeometric representation given by [48, (14.3.1)]

$$\mathsf{P}_\nu^\mu(x) = \left(\frac{1+x}{1-x}\right)^{\frac{1}{2}\mu} {}_2\boldsymbol{F}_1\left(\begin{matrix}-\nu, \nu+1\\ 1-\mu\end{matrix}; \frac{1-x}{2}\right). \tag{40}$$

Let $\nu \in \mathbb{C}$, $\mu \in \mathbb{C}\setminus\mathbb{Z}$, $\nu+\mu \notin -\mathbb{N}$, then a double hypergeometric representation of the Ferrers function of the second kind is given by [48, (14.3.2)]

$$\mathsf{Q}_\nu^\mu(x) = \frac{\pi}{2\sin(\pi\mu)}\Bigg(\cos(\pi\mu)\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}\mu} {}_2\boldsymbol{F}_1\left(\begin{matrix}-\nu, \nu+1\\ 1-\mu\end{matrix}; \frac{1-x}{2}\right) - \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)}\left(\frac{1-x}{1+x}\right)^{\frac{1}{2}\mu} {}_2\boldsymbol{F}_1\left(\begin{matrix}-\nu, \nu+1\\ 1+\mu\end{matrix}; \frac{1-x}{2}\right)\Bigg). \tag{41}$$

Lemma 3. *Let $x \in \mathbb{C}\setminus((-\infty,1]\cup(1,\infty))$, $\lambda,\nu,\mu \in \mathbb{C}$. Then*

$$\mathsf{D}_\lambda^\mu(x) = \frac{\Gamma(\lambda+2\mu)}{2^{\mu-\frac{3}{2}}\sqrt{\pi}\,\Gamma(\mu)\Gamma(\lambda+1)(1-x^2)^{\frac{1}{2}\mu-\frac{1}{4}}}\mathsf{Q}^{\frac{1}{2}-\mu}_{\lambda+\mu-\frac{1}{2}}(x),, \tag{42}$$

such that $\lambda+2\mu \notin -\mathbb{N}_0$ and

$$\mathsf{Q}_\nu^\mu(x) = \frac{\sqrt{\pi}\,\Gamma(\frac{1}{2}-\mu)\Gamma(\nu+\mu+1)}{2^{\mu+1}\Gamma(\nu-\mu+1)(1-x^2)^{\frac{1}{2}\mu}}\mathsf{D}^{\frac{1}{2}-\mu}_{\nu+\mu}(x), \tag{43}$$

such that $\mu \notin \{\frac{1}{2},\frac{3}{2},\ldots\}$ and $\nu+\mu \notin -\mathbb{N}$.

Proof. Start with the definition (36) and use the interrelation between the Gegenbauer function of the second kind on-the-cut and the Ferrers function of the second kind (30). Then applying this relation to the double hypergeometric representation completes the proof. ∎

Theorem 4. *Let $x \in \mathbb{C}\setminus((-\infty,1]\cup(1,\infty))$, $\lambda,\mu \in \mathbb{C}$, such that $\lambda+2\mu \notin -\mathbb{N}_0$. Then*

$$\mathsf{D}_\lambda^\mu(x) = \frac{\sqrt{\pi}}{\cos(\pi\mu)2^{\mu-\frac{1}{2}}\Gamma(\mu)}\Bigg(\frac{\sin(\pi\mu)\Gamma(\lambda+2\mu)}{\Gamma(\lambda+1)(1+x)^{\mu-\frac{1}{2}}}\,{}_2\boldsymbol{F}_1\left(\begin{matrix}\lambda+\mu+\frac{1}{2}, \frac{1}{2}-\lambda-\mu\\ \frac{1}{2}+\mu\end{matrix}; \frac{1-x}{2}\right) - \frac{1}{(1-x)^{\mu-\frac{1}{2}}}\,{}_2\boldsymbol{F}_1\left(\begin{matrix}\lambda+\mu+\frac{1}{2}, \frac{1}{2}-\lambda-\mu\\ \frac{3}{2}-\mu\end{matrix}; \frac{1-x}{2}\right)\Bigg). \tag{44}$$

Proof. Start with the definition (36) and use the interrelation between the Gegenbauer function of the second kind and the Legendre function of the second kind (30). Then comparing with the double hypergeometric representation given by (41) completes the proof. ∎

Note that we also have interrelation between the Ferrers function of the first kind and the Gegenbauer function of the first kind on-the-cut [48, (14.3.21)]

$$\mathsf{P}_\nu^\mu(x) = \frac{\Gamma(\frac{1}{2}-\mu)\Gamma(\nu+\mu+1)}{2^\mu\sqrt{\pi}\,\Gamma(\nu-\mu+1)(1-x^2)^{\frac{1}{2}\mu}}\mathsf{C}^{\frac{1}{2}-\mu}_{\nu+\mu}(x), \tag{45}$$

where $\mu\notin\{\frac12,\frac32,\ldots\}$, $\nu+\mu\notin-\mathbb{N}$, or equivalently

$$\mathsf{C}_\lambda^\mu(x)=\frac{\sqrt{\pi}\,\Gamma(\lambda+2\mu)}{2^{\mu-\frac12}\Gamma(\mu)\Gamma(\lambda+1)(1-x^2)^{\frac12\mu-\frac14}}\mathsf{P}^{\frac12-\mu}_{\lambda+\mu-\frac12}(x). \tag{46}$$

Finally we should add that the Legendre polynomial (the associated Legendre function of the first kind P_ν^μ and the Ferrers function of the first kind P_ν^μ with $\mu=0$ and $\nu=n\in\mathbb{Z}$) is given by [48, (18.7.9)]

$$P_n(x):=P_n^0(x)=\mathsf{P}_n^0(x)=C_n^{\frac12}(x)=P_n^{(0,0)}(x),$$

which vanishes for n negative.

2.2 Brief introduction to Jacobi functions of the first and second kind

Now we will discuss fundamental properties and special values and limits for the Jacobi functions. Jacobi functions are complex solutions $w=w(z)=w_\gamma^{(\alpha,\beta)}(z)$ to the Jacobi differential equation [48, Table 18.8.1]

$$(1-z^2)\frac{\mathrm{d}^2w}{\mathrm{d}z^2}+(\beta-\alpha-z(\alpha+\beta+2))\frac{\mathrm{d}w}{\mathrm{d}z}+\gamma(\alpha+\beta+\gamma+1)w=0, \tag{47}$$

which is a second order linear homogeneous differential equation. Solutions to this differential equation satisfy the following three-term recurrence relation [15, (10.8.11), p. 169]

$$B_\gamma^{(\alpha,\beta)}w_\gamma^{(\alpha,\beta)}(z)+A_\gamma^{(\alpha,\beta)}(z)w_{\gamma+1}^{(\alpha,\beta)}(z)+w_{\gamma+2}^{(\alpha,\beta)}(z)=0, \tag{48}$$

where

$$A_\gamma^{(\alpha,\beta)}(z)=-\frac{(\alpha+\beta+2\gamma+3)\left(\alpha^2-\beta^2+(\alpha+\beta+2\gamma+2)(\alpha+\beta+2\gamma+4)z\right)}{2(\gamma+2)(\alpha+\beta+\gamma+2)(\alpha+\beta+2\gamma+2)}, \tag{49}$$

$$B_\gamma^{(\alpha,\beta)}=\frac{(\alpha+\gamma+1)(\beta+\gamma+1)(\alpha+\beta+2\gamma+4)}{(\gamma+2)(\alpha+\beta+\gamma+2)(\alpha+\beta+2\gamma+2)}. \tag{50}$$

This three-term recurrence relation is very useful for deriving various solutions to (47) when solutions are known for values which have integer separations.

2.2.1 The Jacobi function of the first kind

The Jacobi function of the first kind is a generalization of the Jacobi polynomial (as given by (2)) where the degree is no longer restricted to be an integer. In the following material we derive properties for the Jacobi function of the first kind. In the following result we present the four single Gauss hypergeometric function representations of the Jacobi function of the first kind.

Theorem 5. *Let $\alpha,\beta,\gamma\in\mathbb{C}$ such that $\alpha+\gamma\notin-\mathbb{N}$. Then, the Jacobi function of the first kind $P_\gamma^{(\alpha,\beta)}$: $\mathbb{C}\setminus(-\infty,-1]\to\mathbb{C}$ can be defined by*

$$P_\gamma^{(\alpha,\beta)}(z)=\frac{\Gamma(\alpha+\gamma+1)}{\Gamma(\gamma+1)}\,{}_2\boldsymbol{F}_1\left(\begin{array}{c}-\gamma,\alpha+\beta+\gamma+1\\ \alpha+1\end{array};\frac{1-z}{2}\right) \tag{51}$$

$$=\frac{\Gamma(\alpha+\gamma+1)}{\Gamma(\gamma+1)}\left(\frac{2}{z+1}\right)^{\beta}{}_2\boldsymbol{F}_1\left(\begin{array}{c}-\beta-\gamma,\alpha+\gamma+1\\ \alpha+1\end{array};\frac{1-z}{2}\right) \tag{52}$$

$$=\frac{\Gamma(\alpha+\gamma+1)}{\Gamma(\gamma+1)}\left(\frac{z+1}{2}\right)^{\gamma}{}_2\boldsymbol{F}_1\left(\begin{array}{c}-\gamma,-\beta-\gamma\\ \alpha+1\end{array};\frac{z-1}{z+1}\right) \tag{53}$$

$$=\frac{\Gamma(\alpha+\gamma+1)}{\Gamma(\gamma+1)}\left(\frac{2}{z+1}\right)^{\alpha+\beta+\gamma+1}{}_2\boldsymbol{F}_1\left(\begin{array}{c}\alpha+\gamma+1,\alpha+\beta+\gamma+1\\ \alpha+1\end{array};\frac{z-1}{z+1}\right). \tag{54}$$

Proof. Start with (2) and replace the shifted factorial by a ratio of gamma functions using (7), the factorial $n! = \Gamma(n+1)$ and substitute $n \mapsto \gamma \in \mathbb{C}$, $x \mapsto z$. Application of Euler's transformation (15) and Pfaff's transformation (16) provides the other three single hypergeometric representations. This completes the proof. ■

There exist double Gauss hypergeometric representations of the Jacobi function of the first kind which can be obtained by using the linear transformation formulas for the Gauss hypergeometric function $z \mapsto z^{-1}$, $z \mapsto (1-z)^{-1}$, $z \mapsto 1-z$, $z \mapsto 1-z^{-1}$ [48, (14.3.1)–(14.3.5)], respectively. However, these in general will be given in terms of a sum of two Gauss hypergeometric functions. We will not present the double hypergeometric function representations of the Jacobi function of the first kind here.

One has the following connection relation for the Jacobi function of the first kind.

Corollary 6. *Let $\gamma, \alpha, \beta \in \mathbb{C}$, $z \in \mathbb{C} \setminus (-\infty, 1]$, $\gamma \notin -\mathbb{N}$, $\beta + \gamma \notin \mathbb{N}_0$. Then*

$$P^{(\alpha,\beta)}_{-\gamma-\alpha-\beta-1}(z) = \frac{\Gamma(-\beta-\gamma)\Gamma(\gamma+1)}{\Gamma(-\gamma-\alpha-\beta)\Gamma(\alpha+\gamma+1)} P^{(\alpha,\beta)}_{\gamma}(z). \tag{55}$$

Proof. This connection relation can be derived by using (51) and making the replacement $\gamma \mapsto -\gamma-\alpha-\beta-1$ which leaves the parameters and argument of the hypergeometric function unchanged. Comparing the prefactors completes the proof. ■

Remark 7. *One of the consequences of the definition of the Jacobi function of the first kind is the following special value:*

$$P^{(\alpha,\beta)}_{\gamma}(1) = \frac{\Gamma(\alpha+\gamma+1)}{\Gamma(\alpha+1)\Gamma(\gamma+1)}, \tag{56}$$

where $\alpha + \gamma \notin -\mathbb{N}$. For $\gamma = n \in \mathbb{Z}$ one has

$$P^{(\alpha,\beta)}_{n}(1) = \frac{(\alpha+1)_n}{n!}, \quad P^{(\alpha,\beta)}_{n}(-1) = (-1)^n \frac{(\beta+1)_n}{n!}, \tag{57}$$

which is consistent with (56) *and the parity relation for Jacobi polynomials (see [48, Table 18.6.1]). From* (51) *we have*

$$P^{(\alpha,\beta)}_{0}(z) = 1, \tag{58}$$

and $P^{(\alpha,\beta)}_{k}(z) = 0$ for all $k \in -\mathbb{N}$.

2.2.2 The Jacobi function of the second kind

The Jacobi function of the second kind $Q^{(\alpha,\beta)}_{\gamma}(z)$, $\gamma \in \mathbb{C}$ is a generalization of the Jacobi function of the second kind $Q^{(\alpha,\beta)}_{n}(z)$, $n \in \mathbb{N}_0$ (as given by [15, (10.8.18)]), where the degree is no longer restricted to be an integer. In the following material we derive properties for the Jacobi function of the second kind. Below we give the four single Gauss hypergeometric function representations of the Jacobi function of the second kind.

Theorem 8. *Let $\gamma, \alpha, \beta, z \in \mathbb{C}$ such that $z \in \mathbb{C} \setminus [-1, 1]$, $\alpha+\gamma, \beta+\gamma \notin -\mathbb{N}$. Then, the Jacobi function of the second kind has the following Gauss hypergeometric representations*

$$Q_\gamma^{(\alpha,\beta)}(z) := \frac{2^{\alpha+\beta+\gamma}\Gamma(\alpha+\gamma+1)\Gamma(\beta+\gamma+1)}{(z-1)^{\alpha+\gamma+1}(z+1)^\beta} {}_2\boldsymbol{F}_1\left(\begin{matrix}\gamma+1, \alpha+\gamma+1\\ \alpha+\beta+2\gamma+2\end{matrix}; \frac{2}{1-z}\right) \tag{59}$$

$$= \frac{2^{\alpha+\beta+\gamma}\Gamma(\alpha+\gamma+1)\Gamma(\beta+\gamma+1)}{(z-1)^{\alpha+\beta+\gamma+1}} {}_2\boldsymbol{F}_1\left(\begin{matrix}\beta+\gamma+1, \alpha+\beta+\gamma+1\\ \alpha+\beta+2\gamma+2\end{matrix}; \frac{2}{1-z}\right) \tag{60}$$

$$= \frac{2^{\alpha+\beta+\gamma}\Gamma(\alpha+\gamma+1)\Gamma(\beta+\gamma+1)}{(z-1)^{\alpha}(z+1)^{\beta+\gamma+1}} {}_2\boldsymbol{F}_1\left(\begin{matrix}\gamma+1, \beta+\gamma+1\\ \alpha+\beta+2\gamma+2\end{matrix}; \frac{2}{1+z}\right) \tag{61}$$

$$= \frac{2^{\alpha+\beta+\gamma}\Gamma(\alpha+\gamma+1)\Gamma(\beta+\gamma+1)}{(z+1)^{\alpha+\beta+\gamma+1}} {}_2\boldsymbol{F}_1\left(\begin{matrix}\alpha+\gamma+1, \alpha+\beta+\gamma+1\\ \alpha+\beta+2\gamma+2\end{matrix}; \frac{2}{1+z}\right). \tag{62}$$

Proof. Start with [15, (10.8.18)] and let $n \mapsto \gamma \in \mathbb{C}$ and $x \mapsto z$. Application of Pfaff's ($z \mapsto z/(z-1)$) and Euler's ($z \mapsto z$) transformations [48, (15.8.1)] provides the other three representations. This completes the proof. ∎

One has the following connection relation between Jacobi functions of the first kind and Jacobi functions of the second kind.

Corollary 9. *Let $\gamma, \alpha, \beta \in \mathbb{C}$, $z \in \mathbb{C} \setminus (-\infty, 1]$, $\alpha+\gamma, \beta+\gamma \notin -\mathbb{N}$, $\alpha+\beta+2\gamma \notin \mathbb{Z}$. Then*

$$P_\gamma^{(\alpha,\beta)}(z) = \frac{-2\sin(\pi(\beta+\gamma))}{\pi\sin(\pi(\alpha+\beta+2\gamma+1))} \times \left(\sin(\pi\gamma)Q_\gamma^{(\alpha,\beta)}(z) - \sin(\pi(\alpha+\gamma))\frac{\Gamma(\alpha+\gamma+1)\Gamma(\beta+\gamma+1)}{\Gamma(\gamma+1)\Gamma(\alpha+\beta+\gamma+1)}Q_{-\alpha-\beta-\gamma-1}^{(\alpha,\beta)}(z)\right). \tag{63}$$

Proof. This can be derived by starting with (51), applying the linear transformation [48, (15.8.2)] $z \mapsto z^{-1}$ and then comparing twice with Theorem 8. ∎

Remark 10. *Using* (63) *one can see that for $\gamma = n \in \mathbb{N}_0$, that $Q_{-\alpha-\beta-\gamma-1}^{(\alpha,\beta)}(z)$ is a Jacobi polynomial, namely*

$$Q_{-\alpha-\beta-1-n}^{(\alpha,\beta)}(z) = \frac{\Gamma(-\alpha)\Gamma(-\beta)}{2\Gamma(-\alpha-\beta)}\frac{n!(\alpha+\beta+1)_n}{(\alpha+1)_n(\beta+1)_n}P_n^{(\alpha,\beta)}(z)$$

$$= -\frac{\pi}{2}\frac{\sin(\pi(\alpha+\beta))}{\sin(\pi\alpha)\sin(\pi\beta)}\frac{n!\,\Gamma(\alpha+\beta+1+n)}{\Gamma(\alpha+1+n)\Gamma(\beta+1+n)}P_n^{(\alpha,\beta)}(z). \tag{64}$$

Remark 11. *From Theorem 8 one can derive the following special values for $Q_{-1}^{(\alpha,\beta)}(z)$and $Q_0^{(\alpha,\beta)}(z)$, namely*

$$Q_{-1}^{(\alpha,\beta)}(z) = \frac{2^{\alpha+\beta-1}\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)(z-1)^\alpha(z+1)^\beta}, \tag{65}$$

$$Q_0^{(\alpha,\beta)}(z) = \frac{2^{\alpha+\beta}\Gamma(\alpha+1)\Gamma(\beta+1)}{(z+1)^{\alpha+\beta+1}} {}_2\boldsymbol{F}_1\left(\begin{matrix}\alpha+1, \alpha+\beta+1\\ \alpha+\beta+2\end{matrix}; \frac{2}{1+z}\right). \tag{66}$$

Using the three-term recurrence relation (48) *one can derive values of the Jacobi function of the second kind at all negative integer values. For instance, one can derive*

$$Q_{-2}^{(\alpha,\beta)}(z) = \frac{2^{\alpha+\beta-2}\Gamma(\alpha-1)\Gamma(\beta-1)}{\Gamma(\alpha+\beta-1)(z-1)^\alpha(z+1)^\beta}(\alpha-\beta+(\alpha+\beta-2)z), \tag{67}$$

and also expressions for Jacobi functions of the second kind with further negative integer values of γ.

If one examines the Gauss hypergeometric representations presented in Theorem 8 one can see that they are not defined for certain values of γ, α, β since we must avoid $\alpha+\gamma$ and $\beta+\gamma$ being a negative integer. In fact, these singularities are removable and one is able to compute the values of these Jacobi functions. One can evaluate the Jacobi function of the second kind when the parameters α, β, and degree γ is a non-negative integer in the following result, which was inspired by the work in [53].

Theorem 12. *Let $n,a,b\in\mathbb{N}_0$, $z\in\mathbb{C}\setminus[-1,1]$. Then*

$$Q_n^{(a,b)}(z)=\frac{(-1)^{a+n}}{2^{n+1}}\sum_{\substack{k=0\\k\neq n}}^{a+b+2n}\frac{(-2)^k}{(n-k)}\left((z+1)^{n-k}-(z-1)^{n-k}\right)P_k^{(a+n-k,b+n-k)}(z)$$
$$+\frac{(-1)^a}{2}\log\left(\frac{z+1}{z-1}\right)P_n^{(a,b)}(z). \tag{68}$$

Proof. Start with the integral representation for the Jacobi function of the second kind [49, (4.61.1)]

$$Q_\gamma^{(\alpha,\beta)}(z)=\frac{1}{2^{\gamma+1}(z-1)^\alpha(z+1)^\beta}\int_{-1}^{1}\frac{(1-t)^{\alpha+\gamma}(1+t)^{\beta+\gamma}}{(z-t)^{\gamma+1}}\mathrm{d}t, \tag{69}$$

provided $\Re(\alpha+\gamma)$, $\Re(\beta+\gamma)>-1$ [53, (2.5)] and identify $(\gamma,\alpha,\beta)=(n,a,b)\in\mathbb{N}_0^3$. Then consider

$$\begin{aligned}\mu_{n,k}^{(a,b)}(z)&:=\frac{\mathrm{d}^k}{\mathrm{d}z^k}(1-z)^{n+a}(1+z)^{n+b}\\&=(-1)^k2^kk!(1-z)^{a+n-k}(1+z)^{b+n-k}P_k^{(a+n-k,b+n-k)}(z),\end{aligned}$$

where we have used the Rodrigues-type formula for Jacobi polynomials [48, Table 18.5.1]. It is easy to show that

$$(1-t)^{n+a}(1+t)^{n+b}=\sum_{k=0}^{2n+a+b}\mu_{n,k}^{(a,b)}(z)\frac{(t-z)^k}{k!}, \tag{70}$$

and the right-hand side is valid for all $z\in\mathbb{C}$. Now start with (69) and insert (70) into the integrand and perform the integration over $t\in(-1,1)$ using

$$\int_{-1}^{1}(z-t)^{k-n-1}\,\mathrm{d}t=\begin{cases}\dfrac{(z+1)^{k-n}-(z-1)^{k-n}}{k-n} & \text{if } k\neq n,\\ \log\left(\dfrac{z+1}{z-1}\right) & \text{if } k=n,\end{cases}$$

which completes the proof. ∎

By using (61) we find that if $|z|\sim 1+\epsilon$ then as $\epsilon\to0^+$ one has the following behavior of the Jacobi function of the second kind near the singularity at $z=1$, namely

$$Q_\gamma^{(\alpha,\beta)}(1+\epsilon)\sim\frac{2^{\alpha-1}\Gamma(\alpha)\Gamma(\beta+\gamma+1)}{\Gamma(\alpha+\beta+\gamma+1)\epsilon^\alpha}, \tag{71}$$

where $\Re\alpha>0$, $\beta+\gamma\notin-\mathbb{N}$. By using (61) we see that as $|z|\to\infty$ one has

$$Q_\gamma^{(\alpha,\beta)}(z)\sim\frac{2^{\alpha+\beta+\gamma}\Gamma(\alpha+\gamma+1)\Gamma(\beta+\gamma+1)}{\Gamma(\alpha+\beta+2\gamma+2)z^{\alpha+\beta+\gamma+1}}, \tag{72}$$

where $\alpha+\gamma+1,\beta+\gamma\notin-\mathbb{N}$.

2.2.3 Jacobi functions of the first and second kind on-the-cut

We now refer to the real segment $(-1,1)$ as the cut and the Jacobi functions of the first and second kind on-the-cut as $\mathsf{P}_\gamma^{(\alpha,\beta)}, \mathsf{Q}_\gamma^{(\alpha,\beta)}$. The natural definitions of these Jacobi functions are due to Durand and can be found in [12, (2.3), (2.4)] (see also [5]). These are given as follows:

$$\mathsf{P}_\gamma^{(\alpha,\beta)}(x) := \frac{i}{\pi}\left(\mathrm{e}^{i\pi\alpha}Q_\gamma^{(\alpha,\beta)}(x+i0) - \mathrm{e}^{-i\pi\alpha}Q_\gamma^{(\alpha,\beta)}(x-i0)\right) = P_\gamma^{(\alpha,\beta)}(x\pm i0), \tag{73}$$

$$\mathsf{Q}_\gamma^{(\alpha,\beta)}(x) := \frac{1}{2}\left(\mathrm{e}^{i\pi\alpha}Q_\gamma^{(\alpha,\beta)}(x+i0) + \mathrm{e}^{-i\pi\alpha}Q_\gamma^{(\alpha,\beta)}(x-i0)\right). \tag{74}$$

Note that the Jacobi function of the first kind on-the-cut (73) is simply an analytic continuation of the Jacobi function of the first kind (see Theorem 5) since the complex-valued function is continuous across the real interval $(-1,1]$. On the other hand, the Jacobi function of the second kind is not an analytic continuation of the Jacobi function of the second kind (see Theorem 8). This is because $Q_\gamma^{(\alpha,\beta)}$ is not continuous across the real interval $(-1,1)$. Hence, an 'average' (74) must be taken of the function values with infinitesimal positive and negative arguments in order to define it. Originally, in Szegő's book [49, §4.62.9] (see also [15, (10.8.22)]) a definition for the Jacobi function of the second kind on-the-cut was given by $\mathsf{Q}_\gamma^{(\alpha,\beta)}(x) := \frac{1}{2}\left(Q_\gamma^{(\alpha,\beta)}(x+i0) + Q_\gamma^{(\alpha,\beta)}(x-i0)\right)$, but as is pointed out by Durand [12], Szegő's definition destroys the analogy between $\mathsf{P}_\gamma^{(\alpha,\beta)}(\cos\theta)$, $\mathsf{Q}_\gamma^{(\alpha,\beta)}(\cos\theta)$ and the trigonometric functions. Hence with the updated Durand definitions for the Jacobi functions of the first and second kind on-the-cut (73), (74), one has the following asymptotics as $n\to\infty$, namely [12, p. 77]

$$Q_n^{(\alpha,\beta)}(\cos\theta\pm i0) \sim \frac{1}{2}\left(\frac{\pi}{n}\right)^{\frac{1}{2}}\left(\sin(\tfrac{1}{2}\theta)\right)^{-\alpha-\frac{1}{2}}\left(\cos(\tfrac{1}{2}\theta)\right)^{-\beta-\frac{1}{2}}\mathrm{e}^{\mp iN\theta\mp i\frac{\pi}{2}(\alpha+\frac{1}{2})},$$

where $N := n + \frac{1}{2}\alpha + \frac{1}{2}\beta + \frac{1}{2}$.

There are many double hypergeometric representations of the Jacobi function of the second kind on-the-cut $\mathsf{Q}_\gamma^{(\alpha,\beta)} : \mathbb{C}\setminus((-\infty,1]\cup[1,\infty)) \to \mathbb{C}$. These hypergeometric representations follow by applying the definition (74) to Theorem 8 which provides the Gauss hypergeometric representations for the Jacobi function of the second kind. The application of (74) takes the argument of the Gauss hypergeometric functions just above and below the ray $(1,\infty)$ in which it is known that the Gauss hypergeometric function is discontinuous. The values of the Gauss hypergeometric function z above and below this ray may then be transformed into a region where the Gauss hypergeometric function is continuous in a complex neighborhood of the argument of the Gauss hypergeometric function by utilizing the transformations which one can find in [11, Appendix B]. These transformations, which map from Gauss hypergeometric functions with argument $x\pm i0$ to sums of Gauss hypergeometric functions with arguments given by $1/x$, $1-x$, $1-x^{-1}$ and $(1-x)^{-1}$. Eight Gauss hypergeometric function representations of the Jacobi function of the second kind on-the-cut can be obtained by starting with (59)-(62), applying the transformation [11, Theorem B.1] $z\mapsto z^{-1}$ and by either utilizing the Euler (15) or Pfaff (16) transformations as needed. There are certainly more Gauss hypergeometric representations that can be obtained for the Jacobi function of the second kind on-the-cut by applying [11, Theorems B.2–B.4], but the derivation of these representations must be left to a later publication. We will give two of these here for $\gamma,\alpha,\beta\in\mathbb{C}$ such that $\alpha,\beta\notin\mathbb{Z}$, $\alpha+\gamma,\beta+\gamma\notin-\mathbb{N}$, namely

$$\mathsf{Q}_\gamma^{(\alpha,\beta)}(x)=\frac{\pi}{2\sin(\pi\alpha)}\left(-\cos(\pi\alpha)\frac{\Gamma(\alpha+\gamma+1)}{\Gamma(\gamma+1)}\,{}_2\boldsymbol{F}_1\left(\begin{matrix}-\gamma,\alpha+\beta+\gamma+1\\1+\alpha\end{matrix};\frac{1-x}{2}\right)\right.$$
$$\left.+\frac{\Gamma(\beta+\gamma+1)}{\Gamma(\alpha+\beta+\gamma+1)}\left(\frac{2}{1-x}\right)^\alpha\left(\frac{2}{1+x}\right)^\beta{}_2\boldsymbol{F}_1\left(\begin{matrix}-\alpha-\beta-\gamma,\gamma+1\\1-\alpha\end{matrix};\frac{1-x}{2}\right)\right) \tag{75}$$

$$=\frac{\pi}{2^{\gamma+1}\sin(\pi\alpha)}\left(-\cos(\pi\alpha)\frac{\Gamma(\alpha+\gamma+1)}{\Gamma(\gamma+1)}(1+x)^\gamma\,{}_2\boldsymbol{F}_1\left(\begin{matrix}-\gamma,-\beta-\gamma\\1+\alpha\end{matrix};\frac{x-1}{x+1}\right)\right.$$
$$\left.+\frac{\Gamma(\beta+\gamma+1)}{\Gamma(\alpha+\beta+\gamma+1)}\frac{(1+x)^{\alpha+\gamma}}{(1-x)^\alpha}\,{}_2\boldsymbol{F}_1\left(\begin{matrix}-\alpha-\beta-\gamma,-\alpha-\gamma\\1-\alpha\end{matrix};\frac{x-1}{x+1}\right)\right). \tag{76}$$

Just as we were able to compute the values of the Jacobi function of the second kind with non-negative integer parameters and degree, the same evaluation can be accomplished for the Jacobi function of the second kind on-the-cut which we present now.

Theorem 13. *Let $n,a,b\in\mathbb{N}_0$, $x\in\mathbb{C}\setminus(-\infty,1]\cup[1,\infty)$. Then*

$$\mathsf{Q}_n^{(a,b)}(x)=\frac{(-1)^n}{2^{n+1}}\sum_{\substack{k=0\\k\neq n}}^{a+b+2n}\frac{(-2)^k}{(n-k)}\left((1+x)^{n-k}-(x-1)^{n-k}\right)P_k^{(a+n-k,b+n-k)}(x)+\frac12\log\left(\frac{1+x}{1-x}\right)P_n^{(a,b)}(x). \tag{77}$$

Proof. Start with Theorem 12 and use the definition (74) which completes the proof. ∎

Note that by setting $a=b$ in the above result we can obtain an interesting finite sum expression for the Ferrers functions of the second kind with non-negative integer degree and order given as a sum over ultraspherical polynomials.

Corollary 14. *Let $n,a\in\mathbb{N}_0$, Let $n,a\in\mathbb{N}_0$, $x\in\mathbb{C}\setminus(-\infty,1]\cup[1,\infty)$. Then. Then*

$$\mathsf{Q}_n^a(x)=\frac{(-1)^a(1-x^2)^{\frac12 a}}{2\sqrt{\pi}}\Bigg((-1)^{n+a}2^n(n+a)!$$
$$\times\sum_{\substack{k=0\\k\neq n-a}}^{2n}\frac{(-1)^k\Gamma(n-k+\frac12)}{2^k(2n-k)!(n-a-k)}\left((1+x)^{n-a-k}-(x-1)^{n-a-k}\right)C_k^{n-k+\frac12}(x)$$
$$+2^a\Gamma(a+\tfrac12)\log\left(\frac{1+x}{1-x}\right)C_{n-a}^{a+\frac12}(x)\Bigg). \tag{78}$$

Proof. Start with (13) and set $a=b$. Then utilizing (100) below with (80) completes the proof. ∎

By using (75) we see that as $x=1-\epsilon$ one has as $\epsilon\to0^+$,

$$\mathsf{Q}_\gamma^{(\alpha,\beta)}(1-\epsilon)\sim\frac{2^{\alpha-1}\Gamma(\alpha)\Gamma(\beta+\gamma+1)}{\Gamma(\alpha+\beta+\gamma+1)\epsilon^\alpha}, \tag{79}$$

where $\beta+\gamma+1\notin-\mathbb{N}_0$ and $\Re\alpha>0$.

2.3 Specializations to Gegenbauer, associated Legendre and Ferrers functions

Here we discuss some limiting cases where the Jacobi functions reduce to more elementary functions such as Gegenbauer, associated Legendre, and Ferrers functions.

These identities involve symmetric and antisymmetric Jacobi functions of the first kind. The relation between the symmetric Jacobi function of the first kind and the Gegenbauer function of the first kind for $z \in \mathbb{C} \setminus (-\infty, -1]$ is given by

$$P_\gamma^{(\alpha,\alpha)}(z) = \frac{\Gamma(2\alpha+1)\Gamma(\alpha+\gamma+1)}{\Gamma(\alpha+1)\Gamma(2\alpha+\gamma+1)} C_\gamma^{\alpha+\frac{1}{2}}(z). \tag{80}$$

This follows by starting with (51) and then comparing it to the Gauss hypergeometric representation of the Gegenbauer function of the first kind on the right-hand side using (21).

Remark 15. *The relation between the symmetric Jacobi function of the first kind and the Ferrers function of the first kind is*

$$P_\gamma^{(\alpha,\alpha)}(x) = \frac{2^\alpha \Gamma(\alpha+\gamma+1)}{\Gamma(\gamma+1)(1-x^2)^{\frac{1}{2}\alpha}} \mathsf{P}_{\alpha+\gamma}^{-\alpha}(x), \tag{81}$$

where $x \in \mathbb{C} \setminus ((-\infty, -1] \cup [1, \infty))$ and the relation between the symmetric Jacobi function of the first kind and the associated Legendre function of the first kind is

$$P_\gamma^{(\alpha,\alpha)}(z) = \frac{2^\alpha \Gamma(\alpha+\gamma+1)}{\Gamma(\gamma+1)(z^2-1)^{\frac{1}{2}\alpha}} P_{\alpha+\gamma}^{-\alpha}(z) \tag{82}$$

where $z \in \mathbb{C} \setminus (-\infty, 1]$. These are easily obtained through [48, (18.7.2)] and [48, (14.3.21), (14.3.22)].

Remark 16. *The relation between the antisymmetric Jacobi function of the first kind on-the-cut and the Ferrers function of the first kind and the Gegenbauer function of the first kind on-the-cut is*

$$\begin{aligned} \mathsf{P}_\gamma^{(\alpha,-\alpha)}(x) &= \frac{\Gamma(\alpha+\gamma+1)}{\Gamma(\gamma+1)} \left(\frac{1+x}{1-x}\right)^{\frac{1}{2}\alpha} \mathsf{P}_\gamma^{-\alpha}(x) \\ &= \frac{\Gamma(2\alpha+1)\Gamma(\gamma-\alpha+1)}{2^\alpha \Gamma(\gamma+1)\Gamma(\alpha+1)} (1+x)^\alpha \mathsf{C}_{\gamma-\alpha}^{\alpha+\frac{1}{2}}(x), \end{aligned} \tag{83}$$

where $x \in \mathbb{C} \setminus ((-\infty, -1] \cup [1, \infty))$ and the relation between the antisymmetric Jacobi function of the first kind and the associated Legendre and Gegenbauer function of the first kinds is

$$\begin{aligned} P_\gamma^{(\alpha,-\alpha)}(z) &= \frac{\Gamma(\alpha+\gamma+1)}{\Gamma(\gamma+1)} \left(\frac{z+1}{z-1}\right)^{\frac{1}{2}\alpha} P_\gamma^{-\alpha}(z) \\ &= \frac{\Gamma(2\alpha+1)\Gamma(\gamma-\alpha+1)}{2^\alpha \Gamma(\gamma+1)\Gamma(\alpha+1)} (z+1)^\alpha C_{\gamma-\alpha}^{\alpha+\frac{1}{2}}(z), \end{aligned} \tag{84}$$

where $z \in \mathbb{C} \setminus (-\infty, 1]$. These are obtained by comparing (51) *with* (40) *and* (24).

Remark 17. *One has the following quadratic transformations for the symmetric Jacobi functions of the first kind which can be found in [49, Theorem 4.1]. Let $z \in \mathbb{C} \setminus (-\infty, 1]$, $\gamma, \alpha \in \mathbb{C}$, $\alpha+\gamma \notin -\mathbb{N}$. Then*

$$P_{2\gamma}^{(\alpha,\alpha)}(z) = \frac{\sqrt{\pi}\,\Gamma(\alpha+2\gamma+1)}{2^{2\gamma}\Gamma(\gamma+\frac{1}{2})\Gamma(\alpha+\gamma+1)} P_\gamma^{(\alpha,-\frac{1}{2})}(2z^2-1), \tag{85}$$

where $\alpha+2\gamma\notin-\mathbb{N}$, $\gamma\notin-\mathbb{N}+\frac{1}{2}$, *and*

$$P_{2\gamma+1}^{(\alpha,\alpha)}(z)=\frac{\sqrt{\pi}\,\Gamma(\alpha+2\gamma+2)z}{2^{2\gamma+1}\Gamma(\gamma+\frac{3}{2})\Gamma(\alpha+\gamma+1)}P_{\gamma}^{(\alpha,\frac{1}{2})}(2z^2-1), \tag{86}$$

where $\alpha+2\gamma+1\notin-\mathbb{N}$, $\gamma\notin-\mathbb{N}-\frac{1}{2}$. *The restrictions on the parameters come directly by applying the restrictions on the parameters in Theorem 5 to the Jacobi functions of the first kind on both sides of the relations.*

Below we present some identities which involve symmetric and antisymmetric Jacobi functions of the second kind

Theorem 18. *Two equivalent relations between the symmetric Jacobi function of the second kind and the associated Legendre function of the second kind are given by*

$$Q_{\gamma}^{(\alpha,\alpha)}(z)=\frac{2^{\alpha}\mathrm{e}^{i\pi\alpha}\Gamma(\alpha+\gamma+1)}{\Gamma(\gamma+1)(z^2-1)^{\frac{1}{2}\alpha}}Q_{\alpha+\gamma}^{-\alpha}(z), \tag{87}$$

$$Q_{\gamma}^{(\alpha,\alpha)}(z)=\frac{2^{\alpha}\mathrm{e}^{-i\pi\alpha}\Gamma(\alpha+\gamma+1)}{\Gamma(2\alpha+\gamma+1)(z^2-1)^{\frac{1}{2}\alpha}}Q_{\alpha+\gamma}^{\alpha}(z), \tag{88}$$

where $\alpha+\gamma\notin-\mathbb{N}$. *Also, two equivalent relations between antisymmetric Jacobi functions of the second kind and the associated Legendre function of the second kind are given by*

$$Q_{\gamma}^{(\alpha,-\alpha)}(z)=\frac{\mathrm{e}^{-i\pi\alpha}\Gamma(\gamma-\alpha+1)}{\Gamma(\gamma+1)}\left(\frac{z+1}{z-1}\right)^{\frac{1}{2}\alpha}Q_{\gamma}^{\alpha}(z), \tag{89}$$

$$Q_{\gamma}^{(-\alpha,\alpha)}(z)=\frac{\mathrm{e}^{-i\pi\alpha}\Gamma(\gamma-\alpha+1)}{\Gamma(\gamma+1)}\left(\frac{z-1}{z+1}\right)^{\frac{1}{2}\alpha}Q_{\gamma}^{\alpha}(z), \tag{90}$$

where $\gamma-\alpha\notin-\mathbb{N}$.

Proof. By comparing (59) and (62) with (26) and by using the Legendre duplication formula [48, (5.5.5)] one can obtain all these formulas in a straightforward way. ■

See [9, Section 3, (A.14)] for an interesting application of the symmetric relation for associated Legendre functions of the second kind.

Remark 19. *Observe that by identifying* (87) *and* (88) *and for* $z\in\mathbb{C}\setminus[-1,1]$ *one has*

$$Q_{\gamma}^{(\alpha,\alpha)}(z)=\frac{2^{2\alpha}}{(z^2-1)^{\alpha}}\begin{cases}Q_{\gamma+2\alpha}^{(-\alpha,-\alpha)}(z), & \text{if } z\in\mathbb{C}\setminus[-1,1] \text{ such that } \Re z\ge 0,\\ \mathrm{e}^{2\pi i\alpha}Q_{\gamma+2\alpha}^{(-\alpha,-\alpha)}(z), & \text{if } z\in\mathbb{C}\setminus[-1,1] \text{ such that } \Re z<0 \text{ and } \Im z<0,\\ \mathrm{e}^{-2\pi i\alpha}Q_{\gamma+2\alpha}^{(-\alpha,-\alpha)}(z), & \text{if } z\in\mathbb{C}\setminus[-1,1] \text{ such that } \Re z<0 \text{ and } \Im z\ge 0,\end{cases}$$

where the principal branches of complex powers are taken.

Theorem 20. *Let* $\alpha,\gamma\in\mathbb{C}$, $z\in\mathbb{C}\setminus[-1,1]$, $\alpha+\gamma\notin-\mathbb{N}$. *Then the relations between the symmetric and antisymmetric Jacobi functions of the second kind to the Gegenbauer function of the second kind is given by*

$$Q_{\gamma}^{(\alpha,\alpha)}(z)=\mathrm{e}^{-i\pi(\alpha+\frac{1}{2})}\sqrt{\pi}\,2^{2\alpha}\frac{\Gamma(\alpha+\frac{1}{2})\Gamma(\alpha+\gamma+1)}{\Gamma(2\alpha+\gamma+1)}D_{\gamma}^{\alpha+\frac{1}{2}}(z), \tag{91}$$

where $\alpha\in\mathbb{C}\setminus\{-\frac{1}{2},-\frac{3}{2},-\frac{5}{2},\dots\}$ *and*

$$Q_{\gamma}^{(\alpha,-\alpha)}(z)=\mathrm{e}^{i\pi(\alpha-\frac{1}{2})}2^{2\gamma-\alpha+1}\frac{\Gamma(\alpha+\gamma+1)\Gamma(\frac{1}{2}-\alpha)\Gamma(\gamma+\frac{3}{2})}{\Gamma(2\gamma+2)(z-1)^{\alpha}}D_{\alpha+\gamma}^{\frac{1}{2}-\alpha}(z), \tag{92}$$

where $\alpha\in\mathbb{C}\setminus\{\frac{1}{2},\frac{3}{2},\frac{5}{2},\dots\}$, $\gamma\in\mathbb{C}\setminus\{-\frac{3}{2},-\frac{5}{2},-\frac{7}{2},\dots\}$.

Proof. Start with the definition of the Jacobi function of the second kind (59) and take $\beta = \alpha$. Then comparing (23) using Euler's ($z \mapsto z$) transformation [48, (15.8.1)] produces (91). In order to produce (92), start with (59) and take $\beta = -\alpha$. Then compare (23) using Euler's ($z \mapsto z$) transformation [48, (15.8.1)]. This completes the proof. ■

One has the following quadratic transformations for symmetric Jacobi functions of the second kind.

Theorem 21. *Let $z \in \mathbb{C} \setminus [-1,1]$, $\gamma, \alpha \in \mathbb{C}$, $\alpha + \gamma \notin -\mathbb{N}$. Then*

$$Q_{2\gamma}^{(\alpha,\alpha)}(z) = \frac{\sqrt{\pi}\,\Gamma(\alpha+2\gamma+1)}{2^{2\gamma}\Gamma(\gamma+\frac12)\Gamma(\alpha+\gamma+1)} Q_\gamma^{(\alpha,-\frac12)}(2z^2-1), \tag{93}$$

where $\alpha + 2\gamma \notin -\mathbb{N}$, $\gamma \notin -\mathbb{N} + \frac12$, and

$$Q_{2\gamma+1}^{(\alpha,\alpha)}(z) = \frac{\sqrt{\pi}\,\Gamma(\alpha+2\gamma+2)z}{2^{2\gamma+1}\Gamma(\gamma+\frac32)\Gamma(\alpha+\gamma+1)} Q_\gamma^{(\alpha,\frac12)}(2z^2-1), \tag{94}$$

where $\alpha + 2\gamma + 1 \notin -\mathbb{N}$, $\gamma \notin -\mathbb{N} - \frac12$.

Proof. Starting with the left-hand sides of (93), (94) using the Gauss hypergeometric definition (59), the ${}_2F_1$'s become of a form where $c = 2b$. Then for both equations we use the quadratic transformation of the Gauss hypergeometric function [48, (15.8.14)]. This transforms the ${}_2F_1$ to a form which is recognizable with the right-hand sides through (62), (59), respectively. This completes the proof. The restrictions on the parameters come directly by applying the restrictions on the parameters in Theorem 8 to the Jacobi functions of the second kind on both sides of the relations. ■

There is also an interesting alternative additional quadratic transformation for the Jacobi function of the second kind with $\alpha = \pm\frac12$. Note that there does not seem to be a corresponding formula for the Jacobi function of the first kind since in this case the functions which would appear on the left-hand side would be a sum of two Gauss hypergeometric functions.

Theorem 22. *Let $x \in \mathbb{C}\setminus(-\infty,1] \cup [1,\infty)$, $\beta, \gamma \in \mathbb{C}$ such that $\beta + \gamma + \frac12 \notin -\mathbb{N}_0$. Then*

$$C_{2\gamma+1}^{\beta}(x) = \frac{2^{2\gamma+2}\Gamma(\beta+\gamma+\frac12)}{\Gamma(-\gamma-\frac12)\Gamma(2\gamma+2)\Gamma(\beta)(1-x^2)^{\beta+\gamma+\frac12}} Q_{-\gamma-1}^{(-\frac12,\beta+2\gamma+1)}\left(\frac{1+x^2}{1-x^2}\right), \tag{95}$$

$$C_{2\gamma}^{\beta}(x) = \frac{2^{2\gamma+1}\Gamma(\beta+\gamma+\frac12)x}{\Gamma(-\gamma+\frac12)\Gamma(2\gamma+1)\Gamma(\beta)(1-x^2)^{\beta+\gamma+\frac12}} Q_{-\gamma-1}^{(\frac12,\beta+2\gamma)}\left(\frac{1+x^2}{1-x^2}\right). \tag{96}$$

Proof. The results are easily verified by starting with (59), (61), substituting the related values in the Jacobi function of the second kind and comparing with associated Legendre functions of the first kind with argument $\sqrt{(z-1)/(z+1)}$ and utilizing a quadratic transformation of the Gauss hypergeometric function which relates the two completes the proof. ■

Remark 23. *Note that in Theorem 22, if the argument of the Jacobi function of the second kind has modulus greater than unity then the argument of the Gegenbauer function of the first kind has modulus less than unity.*

Corollary 24. *Let $z, \beta, \gamma \in \mathbb{C}$ such that $z \in \mathbb{C} \setminus [-1,1]$. Then*

$$Q_\gamma^{(\frac12,\beta)}(z) = \frac{2^{\beta+3\gamma+\frac52}\Gamma(-2\gamma-1)\Gamma(\gamma+\frac32)\Gamma(\beta+2\gamma+2)}{\Gamma(\beta+\gamma+\frac32)(z-1)^{\frac12}(z+1)^{\beta+\gamma+1}} C_{-2\gamma-2}^{\beta+2\gamma+2}\left(\sqrt{\frac{z-1}{z+1}}\right), \tag{97}$$

where $-2\gamma-1, \gamma+\frac32, \beta+2\gamma+2 \notin -\mathbb{N}_0$, and

$$Q_\gamma^{(-\frac12,\beta)}(z) = \frac{2^{\beta+3\gamma+\frac12}\Gamma(-2\gamma)\Gamma(\gamma+\frac12)\Gamma(\beta+2\gamma+1)}{\Gamma(\beta+\gamma+\frac12)(z+1)^{\beta+\gamma+\frac12}} C_{-2\gamma-1}^{\beta+2\gamma+1}\left(\sqrt{\frac{z-1}{z+1}}\right), \tag{98}$$

where $-2\gamma, \gamma+\frac12, \beta+2\gamma+1 \notin -\mathbb{N}_0$.

Proof. Inverting Theorem 22 completes the proof. ■

Note that the above results imply the following corollary.

Corollary 25. *Let $z, \beta, \gamma \in \mathbb{C}$ such that $z \in \mathbb{C} \setminus [-1, 1]$, $\gamma + \frac{3}{2}, \beta + \gamma + 1 \notin -\mathbb{N}_0$. Then*

$$Q_\gamma^{(\frac{1}{2},\beta)}(z) = \frac{\Gamma(\gamma + \frac{3}{2})\Gamma(\beta + \gamma + 1)}{\Gamma(\gamma + 1)\Gamma(\beta + \gamma + \frac{3}{2})} \left(\frac{2}{z-1}\right)^{\frac{1}{2}} Q_{\gamma+\frac{1}{2}}^{(-\frac{1}{2},\beta)}(z). \tag{99}$$

Proof. Equating the two relations in Theorem 22 completes the proof. ■

Theorem 26. *Let $x \in \mathbb{C} \setminus ((-\infty, -1] \cup [1, \infty))$. Then the relation between the symmetric and antisymmetric Jacobi functions of the second kind on-the-cut and the Ferrers function of the second kind are given by*

$$\mathsf{Q}_\gamma^{(\alpha,\alpha)}(x) = \frac{2^\alpha \Gamma(\alpha + \gamma + 1)}{\Gamma(\gamma + 1)(1 - x^2)^{\frac{1}{2}\alpha}} \mathsf{Q}_{\gamma+\alpha}^{-\alpha}(x), \tag{100}$$

$$\mathsf{Q}_\gamma^{(\alpha,-\alpha)}(x) = \frac{\Gamma(\alpha + \gamma + 1)}{\Gamma(\gamma + 1)} \left(\frac{1+x}{1-x}\right)^{\frac{1}{2}\alpha} \mathsf{Q}_\gamma^{-\alpha}(x), \tag{101}$$

where $\alpha + \gamma \notin -\mathbb{N}$,

$$\mathsf{Q}_\gamma^{(-\alpha,\alpha)}(x) = \frac{\Gamma(\gamma - \alpha + 1)}{\Gamma(\gamma + 1)} \left(\frac{1-x}{1+x}\right)^{\frac{1}{2}\alpha} \mathsf{Q}_\gamma^{\alpha}(x), \tag{102}$$

where $\gamma - \alpha \notin -\mathbb{N}$.

Proof. The result follows by taking into account (75) and cf. [11, Theorem 3.2]

$$\begin{aligned}\mathsf{Q}_\nu^\mu(x) = \frac{\pi}{2\sin(\pi\mu)} \Bigg[& \cos(\pi(\nu + \mu)) \frac{\Gamma(\nu + \mu + 1)}{\Gamma(\nu - \mu + 1)} \left(\frac{1+x}{1-x}\right)^{\frac{1}{2}\mu} {}_2\boldsymbol{F}_1\left(\begin{matrix}-\nu, \nu + 1 \\ 1 + \mu\end{matrix}; \frac{1+x}{2}\right) \\ & - \cos(\pi\nu) \left(\frac{1-x}{1+x}\right)^{\frac{1}{2}\mu} {}_2\boldsymbol{F}_1\left(\begin{matrix}-\nu, \nu + 1 \\ 1 - \mu\end{matrix}; \frac{1+x}{2}\right) \Bigg],\end{aligned} \tag{103}$$

where $\nu \in \mathbb{C}$, $\mu \in \mathbb{C} \setminus \mathbb{Z}$, such that $\nu + \mu \notin -\mathbb{N}$. The formula (100) is obtained by taking $\beta = \alpha$, then comparing (75) with (103). The other identities follow by applying an analogous method taking $\beta = -\alpha$. This completes the proof. ■

3 Addition theorems for the Jacobi function of the first kind

The Flensted-Jensen–Koornwinder addition theorem for Jacobi functions of the first kind is the extension of the Koornwinder addition theorem for Jacobi polynomials when the degree is allowed to be a complex number. This addition theorem has two separate contexts and some interesting special cases. We will refer to the two separate contexts as the hyperbolic and trigonometric contexts. The hyperbolic context arises when the Jacobi function is analytically continued in the complex plane from the ray $[1, \infty)$. The trigonometric context arises when the argument of the Jacobi function is analytically continued from the real segment $(-1, 1)$. First we will present the addition theorem for the Jacobi function of the first kind in the hyperbolic context. As we will see, the Jacobi function in the trigonometric context can be obtained from the Jacobi functions in the hyperbolic context (and vice versa). We now present the most general form of the addition theorem for Jacobi functions of the first kind in the hyperbolic and trigonometric contexts.

Remark 27. *The addition theorem for Jacobi polynomials was originally derived using group theoretical methods as mentioned in [34] using the representation theory for* $\mathrm{SU}(\alpha+2)/\mathrm{U}(\alpha+1)$ *and* $\mathrm{U}(\alpha+2)/\mathrm{U}(\alpha+1)$ *combined with standard methods discussed, for example, in [50]. The Lie group theoretic setting (and in much more detail for* $\mathrm{SO}(p)\times\mathrm{SO}(q)$*) is carefully described in [30] and is as follows. Let* X *be a compact symmetric group of rank one which is two-point homogeneous for the isometry group* U*. Normalize the metric such that the lengths of the closed geodesics is* 2π*. The choice of a base point in* X *gives an identification of* X *with* U/K*. The function space on* X *decomposes multiplicity free as* $\oplus\mathscr{H}^n$*, and if* d *denotes the geodesic distance between two points* $x,y\in X$*, then the elementary spherical function on* $\mathscr{H}^n$ *is the Jacobi polynomial* $P_n^{(\alpha,\beta)}(\cos d(x,y))$ *with* α,β *suitable 'group values' of the parameters, depending on the choice of* X*. This observation was first made by Élie Cartan in 1929 [6]. For an orthonormal basis* $s_k(x)$ *of* $\mathscr{H}^n$ *it is clear that the real point-pair function* $(x,y)\mapsto\sum_k s_k(x)s_k(y)$ *only depends on the distance* $d(x,y)$*, which in turn implies that*

$$\sum_k s_k(x)\overline{s_k(y)} = c_n P_n^{(\alpha,\beta)}(\cos d(x,y)), \tag{104}$$

for suitable constants c_n*. This identity is the Lie group theoretic meaning of the addition formula for Jacobi polynomials. Picking a suitable basis for* $\mathscr{H}^n$*, this leads to an explicit summation formula for the Jacobi polynomials with an argument depending on* θ_1,θ_2,ϕ*, as a sum. Koornwinder rederived the addition theorem analytically in [32] without relying on group theory, and Flensted-Jensen and Koornwinder extended it to general* n (λ) *and* $\alpha>\beta>-1/2$ *in [17]. In fact, it should be pointed out that there is no group theoretic argument in order to obtain addition theorems such as for Jacobi functions for general values of the parameters. Group theoretic arguments rely on the completeness and orthogonality of the (unitary) representations of the group in question. The case of general parameters may be obtained by appropriate analytic continuation in the group parameters.*

Theorem 28. *Let* $\gamma,\alpha,\beta\in\mathbb{C}$*,* $\gamma\notin\mathbb{Z}$*,* $\alpha,\beta\notin-\mathbb{N}_0$*,* $z_1,z_2\in\mathbb{C}\setminus(-\infty,1]$*,* $x_1,x_2\in\mathbb{C}\setminus((-\infty,-1]\cup[1,\infty))$*,* $x,w\in\mathbb{C}$*,*

$$Z^\pm := Z^\pm(z_1,z_2,w,x) = 2z_1^2z_2^2+2w^2(z_1^2-1)(z_2^2-1)\pm 4z_1z_2wx(z_1^2-1)^{\frac12}(z_2^2-1)^{\frac12}-1, \tag{105}$$

$$\mathsf{X}^\pm := \mathsf{X}^\pm(x_1,x_2,w,x) = 2x_1^2x_2^2+2w^2(1-x_1^2)(1-x_2^2)\pm 4x_1x_2wx(1-x_1^2)^{\frac12}(1-x_2^2)^{\frac12}-1, \tag{106}$$

such that the complex variables $\gamma,\alpha,\beta,z_1,z_2,x_1,x_2,x,w$ *are in some yet to be determined neighborhood of the real line. Then*

$$\begin{aligned}P_\gamma^{(\alpha,\beta)}(Z^\pm) &= \frac{\Gamma(\alpha+1)\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}\sum_{k=0}^\infty \frac{(\alpha+1)_k(\alpha+\beta+\gamma+1)_k}{(\alpha+k)(\beta+1)_k(-\gamma)_k}\\ &\quad\times\sum_{l=0}^k(\mp1)^{k-l}\frac{(\alpha+k+l)(-\beta-\gamma)_l}{(\alpha+\gamma+1)_l}(z_1z_2)^{k-l}\left((z_1^2-1)(z_2^2-1)\right)^{\frac{k+l}{2}}\\ &\qquad\times P_{\gamma-k}^{(\alpha+k+l,\beta+k-l)}(2z_1^2-1)P_{\gamma-k}^{(\alpha+k+l,\beta+k-l)}(2z_2^2-1)w^{k-l}P_l^{(\alpha-\beta-1,\beta+k-l)}(2w^2-1)\frac{\beta+k-l}{\beta}C_{k-l}^\beta(x),\end{aligned} \tag{107}$$

$$\begin{aligned}\mathsf{P}_\gamma^{(\alpha,\beta)}(\mathsf{X}^\pm) &= \frac{\Gamma(\alpha+1)\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}\sum_{k=0}^\infty \frac{(\alpha+1)_k(\alpha+\beta+\gamma+1)_k}{(\alpha+k)(\beta+1)_k(-\gamma)_k}\\ &\quad\times\sum_{l=0}^k(\mp1)^{k-l}\frac{(\alpha+k+l)(-\beta-\gamma)_l}{(\alpha+\gamma+1)_l}(x_1x_2)^{k-l}\left((1-x_1^2)(1-x_2^2)\right)^{\frac{k+l}{2}}\\ &\qquad\times \mathsf{P}_{\gamma-k}^{(\alpha+k+l,\beta+k-l)}(2x_1^2-1)\mathsf{P}_{\gamma-k}^{(\alpha+k+l,\beta+k-l)}(2x_2^2-1)w^{k-l}P_l^{(\alpha-\beta-1,\beta+k-l)}(2w^2-1)\frac{\beta+k-l}{\beta}C_{k-l}^\beta(x).\end{aligned} \tag{108}$$

Proof. Start with the form of the Flensted-Jensen–Koornwinder addition theorem in [17, Theorem 2.1] (see also [40, (24)]). Define the Flensted-Jensen–Koornwinder–Jacobi function of the first kind [17, (2.1)] (Flensted-Jensen–Koornwinder refer to this function as the Jacobi function of the first kind)

$$\varphi_\lambda^{(\alpha,\beta)}(t) := {}_2F_1\left(\begin{matrix}\frac12(\alpha+\beta+1+i\lambda),\frac12(\alpha+\beta+1-i\lambda)\\ \alpha+1\end{matrix};-\sinh^2 t\right), \tag{109}$$

and express it in terms of the Jacobi function of the first kind using

$$\varphi_\lambda^{(\alpha,\beta)}(t) = \frac{\Gamma(\alpha+1)\Gamma(-\frac12(\alpha+\beta-1+i\lambda))}{\Gamma(\frac12(\alpha-\beta+1-i\lambda))}P^{(\alpha,\beta)}_{-\frac12(\alpha+\beta+1+i\lambda)}(\cosh(2t)), \tag{110}$$

which follows by comparing the Gauss hypergeometric representations of the functions. Replacing $\lambda = i(\alpha+\beta+2\gamma+1)$ and setting $z_1=\cosh t_1$, $z_2=\cosh t_2$ and $w=\cos\psi$ produces the form of the addition theorem (107). Then analytically continuing (107) to $\mathsf{X}^\pm\in(-1,1)$ using (73) produces (108). This completes the proof. ■

In the limit as $\gamma\to n\in\mathbb{N}_0$, then Koornwinder's addition theorem for the Jacobi function of the first kind becomes the addition theorem for Jacobi polynomials and the double infinite sum becomes truncated as in (6).

Remark 29. *It is worth mentioning that in the definitions of* $Z^\pm$ (105) *and* $\mathsf{X}^\pm$ (106)*, the influence of the* ±1 *factor on the addition theorems in Theorem 28 and elsewhere in this paper is simply due to the influence of the parity relation for ultraspherical polynomials* (18) *upon the reflection map* $x\mapsto -x$*.*

Remark 30. *Note that there are various ways of expressing the variables* $Z^\pm$ (105) *and* $\mathsf{X}^\pm$ (106)*, which are useful in different applications. For instance, we may also write*

$$\begin{aligned}Z^\pm &= 2z_1^2z_2^2(1-x^2)-1+2(z_1^2-1)(z_2^2-1)\left(w\pm\frac{xz_1z_2}{\sqrt{(z_1^2-1)(z_2^2-1)}}\right)^2\\ &= 2(z_1^2-1)(z_2^2-1)\left(\frac{2z_1^2z_2^2(1-x^2)-1}{2(z_1^2-1)(z_2^2-1)}+\left(w\pm\frac{xz_1z_2}{\sqrt{(z_1^2-1)(z_2^2-1)}}\right)^2\right),\end{aligned}$$

$$\begin{aligned}\mathsf{X}^\pm &= 2x_1^2x_2^2(1-x^2)-1+2(1-x_1^2)(1-x_2^2)\left(w\pm\frac{xx_1x_2}{\sqrt{(1-x_1^2)(1-x_2^2)}}\right)^2\\ &= 2(1-x_1^2)(1-x_2^2)\left(\frac{2x_1^2x_2^2(1-x^2)-1}{2(1-x_1^2)(1-x_2^2)}+\left(w\pm\frac{xx_1x_2}{\sqrt{(1-x_1^2)(1-x_2^2)}}\right)^2\right).\end{aligned}$$

First we will develop some tools which will help us prove the correct form of the double summation addition theorem for the Jacobi function of the second kind. Consider the orthogonality of the ultraspherical polynomials and the Jacobi polynomials with the argument $2w^2-1$.

Lemma 31. *Let* $m,n,p\in\mathbb{N}_0$*,* $\mu\in(-\frac12,\infty)\backslash\{0\}$ $\alpha,\beta\in(-1,\infty)$*,* $\alpha>\beta$*. Then the ultraspherical and Jacobi polynomials satisfy the following orthogonality relations*

$$\int_0^\pi C_m^\mu(\cos\phi)C_n^\mu(\cos\phi)(\sin\phi)^{2\mu}\,\mathrm{d}\phi = \frac{\pi\,\Gamma(2\mu+n)}{2^{2\mu-1}(\mu+n)n!\,\Gamma(\mu)^2}\delta_{m,n}, \tag{111}$$

$$\begin{aligned}&\int_0^1 P_m^{(\alpha-\beta-1,\beta+p)}(2w^2-1)P_n^{(\alpha-\beta-1,\beta+p)}(2w^2-1)w^{2\beta+2p+1}(1-w^2)^{\alpha-\beta-1}\,\mathrm{d}w\\ &\qquad = \frac{\Gamma(\alpha-\beta+n)\Gamma(\beta+1+p+n)}{2(\alpha+p+2n)\Gamma(\alpha+p+n)n!}\delta_{m,n}.\end{aligned} \tag{112}$$

Proof. These orthogonality relations follow easily from [29, (9.8.20), (9.8.2)] upon making the straightforward substitutions. ■

3.1 The parabolic biangle orthogonal polynomial system

Define the 2-variable orthogonal polynomial system which are sometimes referred to as parabolic biangle polynomials [39]

$$\mathcal{P}_{k,l}^{(\alpha,\beta)}(w,\phi) := w^{k-l} P_l^{(\alpha-\beta-1,\beta+k-l)}(2w^2-1) C_{k-l}^{\beta}(\cos\phi), \tag{113}$$

where $k, l \in \mathbb{N}_0$ such that $l \le k$. These 2-variable polynomials are orthogonal over $(w,\phi) \in (0,1)\times(0,\pi)$ with orthogonality measure $\mathrm{d}m^{(\alpha,\beta)}(w,\phi)$ defined by

$$\mathrm{d}m^{(\alpha,\beta)}(w,\phi) := (1-w^2)^{\alpha-\beta-1} w^{2\beta+1} (\sin\phi)^{2\beta}\, \mathrm{d}w\, \mathrm{d}\phi. \tag{114}$$

The orthogonal polynomial system $\mathcal{P}_{k,l}^{(\alpha,\beta)}(w,\phi)$ is deeply connected to the addition theorem for Jacobi functions of the first and second kind. Using the orthogonality relations in Lemma 31 we can derive the orthogonality relation for the 2-variable parabolic biangle polynomials.

Lemma 32. *Let $k, l, k', l' \in \mathbb{N}_0$ such that $l \le k$, $l' \le k'$, $\alpha, \beta \in (-1,\infty)$, $\alpha > \beta$. Then the 2-variable parabolic biangle polynomials satisfy the following orthogonality relation*

$$\int_0^1 \int_0^\pi \mathcal{P}_{k,l}^{(\alpha,\beta)}(w,\phi) \mathcal{P}_{k',l'}^{(\alpha,\beta)}(w,\phi)\, \mathrm{d}m^{(\alpha,\beta)}(w,\phi) = \frac{\pi\,\Gamma(\beta+1+k)\Gamma(2\beta+k-l)\Gamma(\alpha-\beta+l)}{2^{2\beta}\Gamma(\beta)^2(\alpha+k+l)(\beta+k-l)\Gamma(\alpha+k)(k-l)!l!} \delta_{k,k'}\delta_{l,l'}. \tag{115}$$

Proof. Starting with the definition of the 2-variable parabolic biangle polynomials (113) and integrating over $(w,\phi) \in (0,1)\times(0,\pi)$ with measure (114) and using the orthogonality relations in Lemma 31 completes the proof. ■

The following result is a Jacobi function of the first kind generalization of [32, (4.10)] for Jacobi polynomials.

Theorem 33. *Let $k, l \in \mathbb{N}_0$ with $l \le k$, $\gamma, \alpha, \beta \in \mathbb{C}$, $z_1, z_2 \in \mathbb{C}\setminus(-\infty,1]$, $Z^\pm$ defined in* (105)*, such that $x = \cos\phi$ and the complex variables $\gamma, \alpha, \beta, z_1, z_2$ are in some yet to be determined neighborhood of the real line. Then*

$$\begin{aligned}\int_0^1 \int_0^\pi & P_\gamma^{(\alpha,\beta)}(Z^\pm)\, w^{k-l} P_l^{(\alpha-\beta-1,\beta+k-l)}(2w^2-1) C_{k-l}^{\beta}(\cos\phi)\, \mathrm{d}m^{(\alpha,\beta)}(w,\phi) \\ &= (\mp 1)^{k+l} \mathsf{A}_{k,l}^{(\alpha,\beta,\gamma)} (z_1 z_2)^{k-l} ((z_1^2-1)(z_2^2-1))^{\frac{1}{2}(k+l)} P_{\gamma-k}^{(\alpha+k+l,\beta+k-l)}(2z_1^2-1) P_{\gamma-k}^{(\alpha+k+l,\beta+k-l)}(2z_2^2-1),\end{aligned} \tag{116}$$

where

$$\mathsf{A}_{k,l}^{(\alpha,\beta,\gamma)} := \frac{\pi\Gamma(\gamma+1)(\alpha+\beta+\gamma+1)_k \Gamma(2\beta+k-l)\Gamma(\alpha-\beta+l)(-\beta-\gamma)_l}{2^{2\beta}\Gamma(\beta)(-\gamma)_k\,(k-l)!\,l!\,\Gamma(\alpha+\gamma+1+l)}. \tag{117}$$

Proof. Start with the addition theorem for the Jacobi function of the first kind (107) and consider the (k,l)-th term in the double series. It involves a product of two Jacobi functions of the first kind with degree $\gamma-k$ and parameters $(\alpha+k+l, \beta+k-l)$. Replace in (107) the summation indices k, l by k', l', multiply both sides of (107) by $\mathcal{P}_{k,l}^{(\alpha,\beta)}(w,\phi)\,\mathrm{d}m^{(\alpha,\beta)}(w,\phi)$, and integrate both sides over $(w,\phi) \in (0,1)\times(0,\pi)$ using (111), (112). This completes the proof. ■

We will return to the parabolic biangle polynomials in Section 4.

3.2 Special cases of the addition theorem for the Jacobi function of the first kind

In the case when $z_1, z_2, x_1, x_2, w, x = \cos\phi$ are real numbers then the argument of the Jacobi function of the first kind in the addition theorem takes a simpler form convenient form and was proved in Flensted-Jensen–Koornwinder [17].

Remark 34. *In the case where the variables z_1, z_2, x_1, x_2, x, w are real then you may write $Z^\pm$ and $\mathsf{X}^\pm$ as follows*

$$Z^\pm = 2\left| z_1 z_2 \pm \mathrm{e}^{i\phi} w \sqrt{z_1^2-1}\sqrt{z_2^2-1}\right|^2 - 1, \tag{118}$$

$$\mathsf{X}^\pm = 2\left| x_1 x_2 \pm \mathrm{e}^{i\phi} w \sqrt{1-x_1^2}\sqrt{1-x_2^2}\right|^2 - 1. \tag{119}$$

We now give a result which appears to be identical to Theorem 28, but it must be emphasized that it is only in the real case that we are able to write $Z^\pm$, $\mathsf{X}^\pm$ using (118), (119). Otherwise one must use (105), (106).

Theorem 35. *Let $\gamma, \alpha, \beta \in \mathbb{C}$, $\gamma \notin \mathbb{Z}$, $\alpha, \beta \notin -\mathbb{N}_0$, $z_1, z_2 \in (1,\infty)$, $x_1, x_2 \in (-1,1)$, $w \in \mathbb{R}$, $\phi \in [0,\pi]$, and $Z^\pm$, $\mathsf{X}^\pm$ is defined as in (118), (119) respectively. Then*

$$P_\gamma^{(\alpha,\beta)}(Z^\pm) = \frac{\Gamma(\alpha+1)\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} \sum_{k=0}^\infty \frac{(\alpha+1)_k(\alpha+\beta+\gamma+1)_k}{(\alpha+k)(\beta+1)_k(-\gamma)_k}$$
$$\times \sum_{l=0}^k (\mp 1)^{k-l} \frac{(\alpha+k+l)(-\beta-\gamma)_l}{(\alpha+\gamma+1)_l} (z_1 z_2)^{k-l} \left((z_1^2-1)(z_2^2-1)\right)^{\frac{k+l}{2}}$$
$$\times P_{\gamma-k}^{(\alpha+k+l,\beta+k-l)}(2z_1^2-1) P_{\gamma-k}^{(\alpha+k+l,\beta+k-l)}(2z_2^2-1) w^{k-l} P_l^{(\alpha-\beta-1,\beta+k-l)}(2w^2-1) \frac{\beta+k-l}{\beta} C_{k-l}^\beta(\cos\phi), \tag{120}$$

$$\mathsf{P}_\gamma^{(\alpha,\beta)}(\mathsf{X}^\pm) = \frac{\Gamma(\alpha+1)\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} \sum_{k=0}^\infty \frac{(\alpha+1)_k(\alpha+\beta+\gamma+1)_k}{(\alpha+k)(\beta+1)_k(-\gamma)_k}$$
$$\times \sum_{l=0}^k (\mp 1)^{k-l} \frac{(\alpha+k+l)(-\beta-\gamma)_l}{(\alpha+\gamma+1)_l} (x_1 x_2)^{k-l} \left((1-x_1^2)(1-x_2^2)\right)^{\frac{k+l}{2}}$$
$$\times \mathsf{P}_{\gamma-k}^{(\alpha+k+l,\beta+k-l)}(2x_1^2-1) \mathsf{P}_{\gamma-k}^{(\alpha+k+l,\beta+k-l)}(2x_2^2-1) w^{k-l} P_l^{(\alpha-\beta-1,\beta+k-l)}(2w^2-1) \frac{\beta+k-l}{\beta} C_{k-l}^\beta(\cos\phi). \tag{121}$$

Proof. Starting with Theorem 28 and restricting such that the variables $z_1, z_2, x_1, x_2, w, x = \cos\phi$ are real completes the proof. ∎

As in Theorem 28, in the limit as $\gamma \to n \in \mathbb{N}_0$, then Koornwinder's addition theorem for the Jacobi function of the first kind becomes the addition theorem for Jacobi polynomials and the double infinite sum becomes truncated as in (6). Note for all results below, the double or single infinite sums on the right-hand side will always be truncated when the left-hand side is a polynomial (Jacobi polynomial or ultraspherical polynomial). We will not mention this again. Next we have a specialization of Theorem 35 when $w = 1$.

Corollary 36. *Let $\gamma, \alpha, \beta \in \mathbb{C}$, $\gamma \notin \mathbb{Z}$, $\alpha, \beta \notin -\mathbb{N}_0$, $r_1, r_2 \in [0,\infty)$, $\theta_1, \theta_2 \in [0, \frac{\pi}{2}]$, $\phi \in [0,\pi]$,*

$$Z^\pm := 2\left|\cosh r_1 \cosh r_2 \pm \mathrm{e}^{i\phi} \sinh r_1 \sinh r_2\right|^2 - 1 = \cosh(2r_1)\cosh(2r_2) \pm \sinh(2r_1)\sinh(2r_2)\cos\phi, \tag{122}$$

$$\mathsf{X}^\pm := 2\left|\cos\theta_1 \cos\theta_2 \pm \mathrm{e}^{i\phi} \sin\theta_1 \sin\theta_2\right|^2 - 1 = \cos(2\theta_1)\cos(2\theta_2) \pm \sin(2\theta_1)\sin(2\theta_2)\cos\phi. \tag{123}$$

Then

$$P_\gamma^{(\alpha,\beta)}(Z^\pm)=\frac{\Gamma(\alpha+1)\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}\sum_{k=0}^\infty\frac{(\alpha)_k(\frac{\alpha}{2}+1)_k(-\beta-\gamma)_k(\alpha+\beta+\gamma+1)_k}{(\frac{\alpha}{2})_k(\beta+1)_k(-\gamma)_k(\alpha+\gamma+1)_k}(\sinh r_1\sinh r_2)^{2k}$$
$$\times\sum_{l=0}^k\frac{(\mp1)^l(\alpha-\beta)_{k-l}(-\alpha-\gamma-k)_l(-\alpha-2k+1)_l}{(k-l)!(-\alpha-2k)_l(\beta+\gamma+1)_l}(\coth r_1\coth r_2)^l$$
$$\times P_{\gamma-k}^{(\alpha+2k-l,\beta+l)}(\cosh(2r_1))P_{\gamma-k}^{(\alpha+2k-l,\beta+l)}(\cosh(2r_2))\frac{\beta+l}{\beta}C_l^\beta(\cos\phi). \tag{124}$$

$$\mathsf{P}_\gamma^{(\alpha,\beta)}(\mathsf{X}^\pm)=\frac{\Gamma(\alpha+1)\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}\sum_{k=0}^\infty\frac{(\alpha)_k(\frac{\alpha}{2}+1)_k(-\beta-\gamma)_k(\alpha+\beta+\gamma+1)_k}{(\frac{\alpha}{2})_k(\beta+1)_k(-\gamma)_k(\alpha+\gamma+1)_k}(\sin\theta_1\sin\theta_2)^{2k}$$
$$\times\sum_{l=0}^k\frac{(\mp1)^l(\alpha-\beta)_{k-l}(-\alpha-\gamma-k)_l(-\alpha-2k+1)_l}{(k-l)!(-\alpha-2k)_l(\beta+\gamma+1)_l}(\cot\theta_1\cot\theta_2)^l$$
$$\times P_{\gamma-k}^{(\alpha+2k-l,\beta+l)}(\cos(2\theta_1))P_{\gamma-k}^{(\alpha+2k-l,\beta+l)}(\cos(2\theta_2))\frac{\beta+l}{\beta}C_l^\beta(\cos\phi). \tag{125}$$

Proof. Start with Theorem 35 and let $w=1$ using (56) and substituting $l\mapsto l'=k-l$ followed by relabeling $l'\mapsto l$ completes the proof. ∎

By letting $\alpha=\beta$ in Corollary 36 we can relate the above result to associated Legendre and Gegenbauer functions of the first kind. This is mentioned in [34], namely that Koornwinder's addition theorem for Jacobi polynomials generalizes Gegenbauer's addition theorem (4). Similarly, the extension to the Flensted-Jensen–Koornwinder addition theorem for Jacobi functions of the first kind generalizes the addition theorem for Gegenbauer functions of the first kind. First we define the variables

$$\mathcal{Z}^\pm:=\mathcal{Z}^\pm(r_1,r_2,\phi):=\cosh r_1\cosh r_2\pm\sinh r_1\sinh r_2\cos\phi, \tag{126}$$
$$\mathcal{X}^\pm:=\mathcal{X}^\pm(\theta_1,\theta_2,\phi):=\cos\theta_1\cos\theta_2\pm\sin\theta_1\sin\theta_2\cos\phi. \tag{127}$$

Corollary 37. *Let* $\gamma,\alpha\in\mathbb{C}$, $2\alpha\neq1,0,-1,\ldots$, $\gamma\notin-\mathbb{N}$, $r_1,r_2\in[0,\infty)$, $\theta_1,\theta_2\in[0,\frac{\pi}{2}]$, $\phi\in[0,\pi]$, *and* $\mathcal{Z}^\pm$, $\mathcal{X}^\pm$ *as defined in* (126), (127), *respectively. Then*

$$C_\gamma^\alpha(\mathcal{Z}^\pm)=\frac{\Gamma(2\alpha)\Gamma(\gamma+1)}{\Gamma(2\alpha+\gamma)}\sum_{k=0}^\infty\frac{(\mp1)^k\,2^{2k}(\alpha)_k(\alpha)_k}{(-\gamma)_k(2\alpha+\gamma)_k}(\sinh(2r_1)\sinh(2r_2))^k$$
$$\times C_{\gamma-k}^{\alpha+k}(\cosh(2r_1))C_{\gamma-k}^{\alpha+k}(\cosh(2r_2))\frac{\alpha-\frac12+k}{\alpha-\frac12}C_k^{\alpha-\frac12}(\cos\phi), \tag{128}$$

$$C_\gamma^\alpha(\mathcal{X}^\pm)=\frac{\Gamma(2\alpha)\Gamma(\gamma+1)}{\Gamma(2\alpha+\gamma)}\sum_{k=0}^\infty\frac{(\mp1)^k\,2^{2k}(\alpha)_k(\alpha)_k}{(-\gamma)_k(2\alpha+\gamma)_k}(\sin(2\theta_1)\sin(2\theta_2))^k$$
$$\times C_{\gamma-k}^{\alpha+k}(\cos(2\theta_1))C_{\gamma-k}^{\alpha+k}(\cos(2\theta_2))\frac{\alpha-\frac12+k}{\alpha-\frac12}C_k^{\alpha-\frac12}(\cos\phi), \tag{129}$$

or equivalently

$$\frac{1}{(1-\mathcal{Z}^{\pm2})^{\frac12\alpha}}P_\gamma^{-\alpha}(\mathcal{Z}^\pm)=\frac{2^\alpha\Gamma(\alpha+1)}{(\sinh(2\theta_1)\sinh(2\theta_1))^\alpha}$$
$$\times\sum_{k=0}^\infty(\pm1)^k(\alpha-\gamma)_k(\alpha+\gamma+1)_kP_\gamma^{-\alpha-k}(\cosh(2r_1))P_\gamma^{-\alpha-k}(\cosh(2r_2))\frac{\alpha+k}{\alpha}C_k^\alpha(\cos\phi), \tag{130}$$

$$\frac{1}{(1-\mathcal{X}^{\pm 2})^{\frac{1}{2}\alpha}}\mathsf{P}_\gamma^{-\alpha}(\mathcal{X}^\pm)=\frac{2^\alpha\Gamma(\alpha+1)}{(\sin(2\theta_1)\sin(2\theta_1))^\alpha}$$
$$\times\sum_{k=0}^\infty(\pm1)^k(\alpha-\gamma)_k(\alpha+\gamma+1)_k\mathsf{P}_\gamma^{-\alpha-k}(\cos(2\theta_1))\mathsf{P}_\gamma^{-\alpha-k}(\cos(2\theta_2))\frac{\alpha+k}{\alpha}C_k^\alpha(\cos\phi). \tag{131}$$

Proof. Start with Corollary 36 and let $\alpha=\beta$ using (80), (81) respectively for the Jacobi functions of the first kind on the left-hand side and on the right-hand side. Then mapping $(2z_1^2-1,2z_2^2-1)\mapsto(z_1,z_2)$, $(2x_1^2-1,2x_2^2-1)\mapsto(x_1,x_2)$, where $z_1=\cosh r_1$, $z_2=\cosh r_2$, $x_1=\cos\theta_1$, $x_2=\cos\theta_2$, and simplifying using (80)–(82) completes the proof.

Another way to prove this result is to take $\beta=-\frac{1}{2}$, $w=\cos\psi=1$, $\gamma\to2\gamma$ in (120) and use the quadratic transformation (85). After using (80), this produces the left-hand side of (128) with degree 4γ and order given by $\alpha+\frac{1}{2}$. Because we set $w=1$, the sum over l only survives for $l=0,1$. By taking $4\gamma\mapsto\gamma$ and expressing the contribution due to each of these terms one can identify Gegenbauer's addition theorem through repeated application of (80) on the right-hand side and that

$$\sum_{k=0}^\infty(f_{2k}+f_{2k+1})=\sum_{k=0}^\infty f_k,$$

for some sequence $\{f\}_{k\in\mathbb{N}_0}$, one arrives at (128). This other proof is similar for (129). ∎

4 Addition theorems for the Jacobi function of the second kind

Now we present double summation addition theorems for the Jacobi functions of the second kind in the hyperbolic and trigonometric contexts.

Remark 38. *There is no direct group-theoretical derivation for an addition formula for the Jacobi functions of the second kind. However, those functions satisfy the same differential recurrence relations as the functions of the first kind, and the actions of those operators and the Jacobi differential equation* (47) *on functions of either kind for general parameters give realizations of the Lie algebras of the groups considered. One would therefore expect the Jacobi functions of the second kind to satisfy addition formulas with the same structure as those for the functions of the first kind. This is known to hold, for example, for the Gegenbauer functions of the first and second kind [14, §8].*

4.1 The hyperbolic context for the addition theorem for the Jacobi function of the second kind

Define

$$z_{\lessgtr}:=\begin{matrix}\min\\ \max\end{matrix}\{z_1,z_2\}, \tag{132}$$

where $z_1,z_2\in(1,\infty)$, and in the case where $z_1,z_2\in\mathbb{C}$, then if one takes without loss of generality $z_1=z_>$ to lie on an ellipse with foci at ±1, then $z_2=z_<$ must be chosen to be in the interior of that ellipse.

Theorem 39. *Let $\gamma, \alpha, \beta \in \mathbb{C}$, $z_1, z_2 \in \mathbb{C} \setminus (-\infty, 1]$, $x, w \in \mathbb{C}$, $Z^\pm$ defined in* (105)*, such that the complex variables $\gamma, \alpha, \beta, z_1, z_2, x, w$ are in some yet to be determined neighborhood of the real line. Then*

$$Q_\gamma^{(\alpha,\beta)}(Z^\pm) = \frac{\Gamma(\alpha+1)\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} \sum_{k=0}^\infty \frac{(\alpha+1)_k(\gamma+1)_k}{(\alpha+k)(\beta+1)_k(1-\gamma)_k}$$
$$\times \sum_{l=0}^k (\pm 1)^{k-l}(\alpha+k+l)(z_1z_2)^{k-l}\left((z_1^2-1)(z_2^2-1)\right)^{\frac{k+l}{2}}$$
$$\times P_{\gamma-k}^{(\alpha+k+l,\beta+k-l)}(2z_<^2-1)Q_{\gamma-k}^{(\alpha+k+l,\beta+k-l)}(2z_>^2-1)w^{k-l}P_l^{(\alpha-\beta-1,\beta+k-l)}(2w^2-1)\frac{\beta+k-l}{\beta}C_{k-l}^\beta(x). \quad (133)$$

Proof. Start with the addition formula (107) for the Jacobi functions of the first kind in (107). The series was shown to converge by Flensted-Jensen and Koornwinder [17, Theorem 2.1]. Now use the connection relation (63) which relates the Jacobi function of the first kind to two Jacobi functions of the second kind, one on the left-hand side of (63) and again on the right-hand side for the function of the first kind with argument $2z_2^2-1$ on the right, assuming without loss of generality that $z_2 = z_>$. This gives an expression in which the asymptotic behavior of the two terms on the left for $z_2 \to \infty$ matches the term-by-term asymptotic behavior of the two corresponding series on the right, suggesting that they should be identified and that the series converge separately. To exploit this, use the projection of the left- and right-hand sides by the parabolic biangle polynomials, Theorem 33, with the same substitution of the connection formula for the Jacobi functions of the first and second kind ((63) once in the integrand). This results in the following equation

$$\mathsf{B}_\gamma^{(\alpha,\beta)} \int_0^1\int_0^\pi Q_\gamma^{(\alpha,\beta)}(Z^\pm)\, w^{k-l}P_l^{(\alpha-\beta-1,\beta+k-l)}(2w^2-1)C_{k-l}^\beta(\cos\phi)\,\mathrm{d}m^{(\alpha,\beta)}(w,\phi)$$
$$+\mathsf{C}_{\gamma,k,l}^{(\alpha,\beta)}(z_1z_2)^{k-l}((z_1^2-1)(z_2^2-1))^{\frac12(k+l)}P_{\gamma-k}^{(\alpha+k+l,\alpha+k-l)}(2z_<^2-1)Q_{\gamma-k}^{(\alpha+k+l,\alpha+k-l)}(2z_>^2-1)$$
$$= \mathsf{D}_\gamma^{(\alpha,\beta)} \int_0^1\int_0^\pi Q_{-\alpha-\beta-\gamma-1}^{(\alpha,\beta)}(Z^\pm)w^{k-l}P_l^{(\alpha-\beta-1,\beta+k-l)}(2w^2-1)C_{k-l}^\beta(\cos\phi)\,\mathrm{d}m^{(\alpha,\beta)}(w,\phi)$$
$$+\mathsf{E}_{\gamma,k,l}^{(\alpha,\beta)}(z_1z_2)^{k-l}((z_1^2-1)(z_2^2-1))^{\frac12(k+l)}P_{-\alpha-\beta-\gamma-k-1}^{(\alpha+k+l,\alpha+k-l)}(2z_<^2-1)Q_{-\alpha-\beta-\gamma-k-1}^{(\alpha+k+l,\alpha+k-l)}(2z_>^2-1), \quad (134)$$

where

$$\mathsf{B}_\gamma^{(\alpha,\beta)} := \frac{-2\sin(\pi\gamma)\sin(\pi(\beta+\gamma))}{\pi\sin(\pi(\alpha+\beta+2\gamma+1))}, \quad (135)$$

$$\mathsf{C}_{\gamma,k,l}^{(\alpha,\beta)} := \frac{\sin(\pi\gamma)\sin(\pi(\beta+\gamma))\Gamma(\gamma+1)(\alpha+\beta+\gamma+1)_k\Gamma(2\beta+k-l)\Gamma(\alpha-\beta+l)(-\beta-\gamma)_l}{2^{2\beta-1}\sin(\pi(\alpha+\beta+2\gamma+1))\Gamma(\beta)(-\gamma)_k(k-l)!\,l!\,\Gamma(\alpha+\gamma+1+l)}, \quad (136)$$

$$\mathsf{D}_\gamma^{(\alpha,\beta)} := \frac{-2\sin(\pi(\alpha+\gamma))\sin(\pi(\beta+\gamma))\Gamma(\alpha+\gamma+1)\Gamma(\beta+\gamma+1)}{\pi\sin(\pi(\alpha+\beta+2\gamma+1))\Gamma(\gamma+1)\Gamma(\alpha+\beta+\gamma+1)}, \quad (137)$$

$$\mathsf{E}_{\gamma,k,l}^{(\alpha,\beta)} := \frac{\sin(\pi(\alpha+\gamma))\sin(\pi(\beta+\gamma))\Gamma(\beta+\gamma+1)\Gamma(2\beta+k-l)\Gamma(\alpha-\beta+l)}{2^{2\beta-1}\sin(\pi(\alpha+\beta+2\gamma+1))\Gamma(\beta)\Gamma(\alpha+\beta+\gamma+1)(k-l)!\,l!}. \quad (138)$$

Now consider the asymptotics of all four terms as $z_2 \to \infty$. The asymptotic behavior of $Z^\pm$ as $z_2 \to \infty$ is $Z^\pm \sim z_2^2$. The behavior of the Jacobi function of the second kind as the argument $|z| \to \infty$ is (72)

$$Q_\gamma^{(\alpha,\beta)}(z) \sim \frac{1}{z^{\alpha+\beta+\gamma+1}}.$$

Therefore, one has the following asymptotic behavior considered as functions of $\zeta = z_>$ with $z_<$ fixed,

$$Q_\gamma^{(\alpha,\beta)}(Z^\pm) \sim (Z^\pm)^{-\gamma-\alpha-\beta-1} \sim \zeta^{-2\gamma-2\alpha-2\beta-2}, \tag{139}$$

$$Q_{-\alpha-\beta-\gamma-1}^{(\alpha,\beta)}(Z^\pm) \sim (Z^\pm)^\gamma \sim \zeta^{2\gamma}, \tag{140}$$

$$Q_{\gamma-k}^{(\alpha+k+l,\beta+k-l)}(\zeta) \sim \zeta^{-\gamma-\alpha-\beta-k-1}, \tag{141}$$

$$Q_{-\alpha-\beta-\gamma-1-k}^{(\alpha+k+l,\beta+k-l)}(\zeta) \sim \zeta^{\gamma-k}. \tag{142}$$

The above relation (134) with leading order asymptotic contribution as $z_2 \to \infty$ taken out can be written as a function of two analytic functions $\mathsf{f}(z_2^{-1}) := \mathsf{f}_{\gamma,k,l}^{(\alpha,\beta)}(z_1, z_2^{-1})$, $\mathsf{g}(z_2^{-1}) := \mathsf{g}_{\gamma,k,l}^{(\alpha,\beta)}(z_1, z_2^{-1})$, as

$$z_2^{-2(\gamma+\alpha+\beta+1)}\mathsf{f}(z_2^{-1}) = z_2^{2\gamma}\mathsf{g}(z_2^{-1}). \tag{143}$$

For $4\Re\gamma \notin -2(\alpha+\beta+1)+\mathbb{Z}$, the only way the equation can be true is if f and g vanish identically. The case of general γ then follows by analytic continuation in γ. Therefore we have now verified separately all terms in the double series expansion of the Jacobi function of the second kind given in (133), and the corresponding series for $Q_{-\alpha-\beta-\gamma-1}^{(\alpha,\beta)}(Z^\pm)$. The two series do not mix, have different asymptotic behaviors for $z_2 = z_> \to \infty$, and must converge separately given the overall convergence proved by Flensted-Jensen and Koornwinder [17, Theorem 2.1]. This completes the proof. ■

Remark 40. *If you apply* (64) *to the Jacobi functions of the second kind on the left-hand side and right-hand side of* (133)*, then it becomes the hyperbolic context of the addition theorem for Jacobi polynomials.*

Corollary 41. *Let $k, l \in \mathbb{N}_0$ with $l \le k$, $\gamma, \alpha, \beta \in \mathbb{C}$, $z_1, z_2 \in \mathbb{C} \setminus (-\infty, 1]$, $Z^\pm$ defined in* (105)*, such that $x = \cos\phi$ and the complex variables $\gamma, \alpha, \beta, z_1, z_2$ are in some yet to be determined neighborhood of the real line. Then*

$$\int_0^1 \int_0^\pi Q_\gamma^{(\alpha,\beta)}(Z^\pm)\, w^{k-l} P_l^{(\alpha-\beta-1,\beta+k-l)}(2w^2-1) C_{k-l}^\beta(\cos\phi)\, \mathrm{d}m^{(\alpha,\beta)}(w,\phi),$$
$$= (\pm1)^{k+l}\, \mathsf{A}_{k,l}^{(\alpha,\beta,\gamma)}(z_1z_2)^{k-l}((z_1^2-1)(z_2^2-1))^{\frac12(k+l)} P_{\gamma-k}^{(\alpha+k+l,\beta+k-l)}(2z_<^2-1) Q_{\gamma-k}^{(\alpha+k+l,\beta+k-l)}(2z_>^2-1), \tag{144}$$

where $\mathsf{A}_{k,l}^{(\alpha,\beta,\gamma)}$ is defined in (117)*.*

Proof. This follows directly from (143) since in the proof of Theorem 39, we showed that $\mathsf{f} = 0$. The result $\mathsf{g} = 0$ is equivalent to this result under the transformation $\gamma \mapsto -\gamma-\alpha-\beta-1$. ■

Remark 42. *The above integral representations Theorem 33 and Corollary 41 are equivalent to the double summation addition theorems for the Jacobi function of the first kind* (107) *and second kind, Theorem 39.*

Remark 43. *One has the following well-known product representations which are the $k = l = 0$ contribution of the integral representations Theorem 33 and Corollary 41. Let $x = \cos\phi$, $\gamma, \alpha, \beta \in \mathbb{C}$, $z_1, z_2 \in \mathbb{C}\setminus(-\infty, 1]$, $Z^\pm$ defined in* (105)*, such that the complex variables $\gamma, \alpha, \beta, z_1, z_2$ are in some yet to be determined neighborhood of the real line. Then*

$$P_\gamma^{(\alpha,\beta)}(2z_1^2-1) P_\gamma^{(\alpha,\beta)}(2z_2^2-1) = \frac{2\Gamma(\alpha+\gamma+1)}{\sqrt{\pi}\,\Gamma(\gamma+1)\Gamma(\beta+\frac12)\Gamma(\alpha-\beta)} \int_0^1 \int_0^\pi P_\gamma^{(\alpha,\beta)}(Z^\pm)\, \mathrm{d}m^{(\alpha,\beta)}(w,\phi), \tag{145}$$

$$P_\gamma^{(\alpha,\beta)}(2z_<^2-1) Q_\gamma^{(\alpha,\beta)}(2z_>^2-1) = \frac{2\Gamma(\alpha+\gamma+1)}{\sqrt{\pi}\,\Gamma(\gamma+1)\Gamma(\beta+\frac12)\Gamma(\alpha-\beta)} \int_0^1 \int_0^\pi Q_\gamma^{(\alpha,\beta)}(Z^\pm)\, \mathrm{d}m^{(\alpha,\beta)}(w,\phi). \tag{146}$$

For independent verification of these product representations, see [16, Theorem 4.1] for the product formula (145) *and [16, p. 255] for the product formula* (146)*.*

Remark 44. *As pointed out by one of the referees, our proof of the addition theorem for the Jacobi function of the second kind was chosen so that it was the most economical, following as it does from the role of Jacobi functions as spherical functions on symmetric spaces [34] extended to general parameters as in [17]. However, this does not mean that our chosen method is the most insightful method of proof overall. In fact, Flensted-Jensen and Koornwinder (1979) [17], derived the addition theorem for the Jacobi function of the first kind from the product formula* (145) *which the same authors had obtained earlier in [16, Theorem 4.1]. Inspection of the proof of the product formula* (145) *in [16] shows that this really comes from the product formula* (146), *for Jacobi functions of the second kind [16, Proof of Theorem 4.1, p. 252]. In this proof, the authors expand the Jacobi function of the first kind using [16, (4.6)], which is simply the connection formula* (63) *which expresses the Jacobi function of the first kind as a linear combination of two Jacobi functions of the second kind with different degrees. So, as the referee pointed out, a more informative proof of the addition theorem for the Jacobi function of the second kind, Theorem 39, would be to derive it from the product formula* (146) *in the same way as Theorem 28 is derived in [17] from the product formula* (145).

In the case when $z_1, z_2, w, x = \cos\phi$ are real numbers, then the argument of the Jacobi function of the second kind in the addition theorem for the Jacobi function of the second kind takes a simpler and more convenient form. This is analogous to the Flensted-Jensen–Koornwinder addition theorem of the first kind (120). We present this result now.

Theorem 45. *Let $\gamma, \alpha, \beta, w \in \mathbb{R}$, such that $\gamma \notin \mathbb{Z}$, $\alpha \notin -\mathbb{N}$, $\beta \in (-\frac{1}{2}, \infty) \setminus \{0\}$, $\alpha+\gamma, \beta+\gamma \notin -\mathbb{N}$, $\phi \in [0, \pi]$, $z_1, z_2 \in (1, \infty)$, and $Z^\pm$, $z_{\lessgtr}$, as defined as in* (118), (132), *respectively. Then*

$$Q_\gamma^{(\alpha,\beta)}(Z^\pm) = \frac{\Gamma(\alpha+1)\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} \sum_{k=0}^{\infty} \frac{(\alpha+1)_k(\alpha+\beta+\gamma+1)_k}{(\alpha+k)(\beta+1)_k(-\gamma)_k}$$
$$\times \sum_{l=0}^{k} (\pm 1)^{k+l} \frac{(\alpha+k+l)(-\beta-\gamma)_l}{(\alpha+\gamma+1)_l} (z_1 z_2)^{k-l} \left((z_1^2-1)(z_2^2-1)\right)^{\frac{k+l}{2}}$$
$$\times P_{\gamma-k}^{(\alpha+k+l,\beta+k-l)}(2z_<^2-1) Q_{\gamma-k}^{(\alpha+k+l,\beta+k-l)}(2z_>^2-1) w^{k-l} P_l^{(\alpha-\beta-1,\beta+k-l)}(2w^2-1) \frac{\beta+k-l}{\beta} C_{k-l}^{\beta}(\cos\phi). \quad (147)$$

Proof. This follows from Theorem 39 by setting the variables $\gamma, \alpha, \beta, z_1, z_2, w, x = \cos\phi$ to real numbers. ■

4.2 The trigonometric context of the addition theorem for the Jacobi function of the second kind

In the trigonometric context for the addition theorem for Jacobi functions of the second kind, one must then use the Jacobi function of the second kind on-the-cut $\mathsf{Q}_\gamma^{(\alpha,\beta)}(x)$ (74), which are defined in Section 2.2.3 and have a hypergeometric representation given by (75). Note that this representation is not unique and there are many other double Gauss hypergeometric representations of this function. For more about this see the discussion immediately above (75). Define

$$x_{\lessgtr} := \frac{\min}{\max}\{x_1, x_2\}, \quad (148)$$

where $x_1, x_2 \in (-1, 1)$, and in the case where $x_1, x_2 \in \mathbb{C}$, then if one takes without loss of generality $x_1 = x_>$ to lie on an ellipse with foci at ± 1, then $x_2 = x_<$ must be chosen to be in the interior of that ellipse.

Theorem 46. *Let $k, l \in \mathbb{N}_0$, $l \le k$, $\gamma, \alpha, \beta \in \mathbb{C}$, $x_1, x_2 \in \mathbb{C} \setminus ((-\infty, -1] \cup [1, \infty))$, $\alpha \notin \mathbb{Z}$, $\alpha+\gamma, \beta+\gamma \notin -\mathbb{N}$, $\mathsf{X}^\pm$, $x_{\lessgtr}$ as defined in* (106), (148) *respectively, such that the complex variables $\gamma, \alpha, \beta, x_1, x_2$ are in some yet to be determined neighborhood of the real line. Then*

$$\int_0^1 \int_0^\pi \mathsf{Q}_\gamma^{(\alpha,\beta)}(X^\pm)\, w^{k-l} P_l^{(\alpha-\beta-1,\beta+k-l)}(2w^2-1) C_{k-l}^{\beta}(\cos\phi)\, \mathrm{d}m^{(\alpha,\beta)}(w,\phi)$$
$$= (\mp 1)^{k+l} \mathsf{A}_{k,l}^{(\alpha,\beta,\gamma)} (x_1 x_2)^{k-l} ((1-x_1^2)(1-x_2^2))^{\frac{1}{2}(k+l)} \mathsf{Q}_{\gamma-k}^{(\alpha+k+l,\beta+k-l)}(2x_<^2-1) \mathsf{P}_{\gamma-k}^{(\alpha+k+l,\beta+k-l)}(2x_>^2-1), \quad (149)$$

where $\mathsf{A}_{k,l}^{(\alpha,\beta,\gamma)}$ *is defined in* (117).

Proof. Starting with Corollary 41 and directly applying (74) completes the proof. ■

Corollary 47. *Let* $\gamma,\alpha,\beta\in\mathbb{C}$, $x_1,x_2\in\mathbb{C}\setminus((-\infty,-1]\cup[1,\infty))$, $\alpha\notin\mathbb{Z}$, $\alpha+\gamma,\beta+\gamma\notin-\mathbb{N}$, $\mathsf{X}^\pm$, $x_\lessgtr$ *as defined in* (106), (148) *respectively, such that the complex variables* $\gamma,\alpha,\beta,x_1,x_2$ *are in some yet to be determined neighborhood of the real line. Then*

$$\int_0^1\int_0^\pi \mathsf{Q}_\gamma^{(\alpha,\beta)}(X^\pm)\,\mathrm{d}m^{(\alpha,\beta)}(w,\phi)=\frac{\sqrt{\pi}\,\Gamma(\gamma+1)\Gamma(\beta+\frac12)\Gamma(\alpha-\beta)}{2\Gamma(\alpha+\gamma+1)}\mathsf{Q}_\gamma^{(\alpha,\beta)}(2x_<^2-1)\mathsf{P}_\gamma^{(\alpha,\beta)}(2x_>^2-1). \quad (150)$$

Proof. Starting with Theorem 46 and setting $k=l=0$ completes the proof. ■

Theorem 48. *Let* $\gamma,\alpha,\beta\in\mathbb{C}$, $x_1,x_2\in\mathbb{C}\setminus((-\infty,-1]\cup[1,\infty))$, $\alpha\notin\mathbb{Z}$, $\alpha+\gamma,\beta+\gamma\notin-\mathbb{N}$, $x,w\in\mathbb{C}$ *with* $\mathsf{X}^\pm$, $x_\lessgtr$ *as defined in* (106), (148) *respectively, such that the complex variables* $\gamma,\alpha,\beta,x_1,x_2,x,w$ *are in some yet to be determined neighborhood of the real line. Then*

$$\mathsf{Q}_\gamma^{(\alpha,\beta)}(\mathsf{X}^\pm)=\frac{\Gamma(\alpha+1)\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}\sum_{k=0}^\infty\frac{(\alpha+1)_k(\alpha+\beta+\gamma+1)_k}{(\alpha+k)(\beta+1)_k(-\gamma)_k}$$
$$\times\sum_{l=0}^k(\mp1)^{k-l}\frac{(\alpha+k+l)(-\beta-\gamma)_l}{(\alpha+\gamma+1)_l}(x_1x_2)^{k-l}\left((1-x_1^2)(1-x_2^2)\right)^{\frac{k+l}{2}}$$
$$\times\mathsf{Q}_{\gamma-k}^{(\alpha+k+l,\beta+k-l)}(2x_<^2-1)\mathsf{P}_{\gamma-k}^{(\alpha+k+l,\beta+k-l)}(2x_>^2-1)w^{k-l}P_l^{(\alpha-\beta-1,\beta+k-l)}(2w^2-1)\frac{\beta+k-l}{\beta}C_{k-l}^\beta(x). \quad (151)$$

Proof. The result follows by starting with the addition theorem for Jacobi functions of the second kind (147) and applying the definition (74) completes the proof. ■

In the case when $z_1,z_2,w,x=\cos\phi$ are real numbers, then the argument of the Jacobi function of the second kind on-the-cut in the addition theorem for the Jacobi function of the second kind takes a simpler and more convenient form. This is analogous to the addition theorem (120). We present this result now.

Corollary 49. *Let* $\gamma,\alpha,\beta,w\in\mathbb{R}$, *such that* $\gamma\notin\mathbb{Z}$, $\alpha,\alpha+\gamma,\beta+\gamma\notin-\mathbb{N}$, $x_1,x_2\in(-1,1)$, $\phi\in[0,\pi]$, *with* $\mathsf{X}^\pm$, $x_\lessgtr$ *as defined in* (119), (148) *respectively. Then*

$$\mathsf{Q}_\gamma^{(\alpha,\beta)}(\mathsf{X}^\pm)=\frac{\Gamma(\alpha+1)\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}\sum_{k=0}^\infty\frac{(\alpha+1)_k(\alpha+\beta+\gamma+1)_k}{(\alpha+k)(\beta+1)_k(-\gamma)_k}$$
$$\times\sum_{l=0}^k(\mp1)^{k-l}\frac{(\alpha+k+l)(-\beta-\gamma)_l}{(\alpha+\gamma+1)_l}(x_1x_2)^{k-l}\left((1-x_1^2)(1-x_2^2)\right)^{\frac{k+l}{2}}$$
$$\times\mathsf{Q}_{\gamma-k}^{(\alpha+k+l,\beta+k-l)}(2x_<^2-1)\mathsf{P}_{\gamma-k}^{(\alpha+k+l,\beta+k-l)}(2x_>^2-1)w^{k-l}P_l^{(\alpha-\beta-1,\beta+k-l)}(2w^2-1)\frac{\beta+k-l}{\beta}C_{k-l}^\beta(\cos\phi). \quad (152)$$

Proof. This result follows from Theorem 45 by setting the complex variables to be real. ■

4.3 Olver normalized Jacobi functions and their addition theorems

Koornwinder's addition theorem for Jacobi polynomials with degree $n\in\mathbb{N}_0$ (6) is terminating with the k sum being over $k\in\{0,\ldots,n\}$. One can see this by examination of (120), (121) by recognizing that both Jacobi polynomials $P_{\gamma-k}^{(\alpha+k+l,\beta+k-l)}$ vanish for $\gamma=n\in\mathbb{N}_0$ and $k\geq n+1$. However, considering the limit as $\gamma\to n$ for all values of $k\in\mathbb{N}_0$ in Koornwinder's addition theorem, the factor $1/(-\gamma)_k$ blows up for $k\geq n+1$. On the other hand, this factor in the limit when multiplied by the Jacobi function of the first kind prefactor containing $1/\Gamma(\gamma-k+1)$, while considering the residues of the gamma function, the product will be finite, namely

$$\lim_{\gamma\to n}\frac{1}{(-\gamma)_k\Gamma(\gamma-k+1)}=\lim_{\gamma\to n}\frac{\Gamma(-\gamma)}{\Gamma(-\gamma+k)\Gamma(\gamma-k+1)}=\frac{(-1)^k}{n!},\tag{153}$$

for $k\ge n+1$. But when this finite factor is multiplied by the second Jacobi polynomial $P_{\gamma-k}^{(\alpha+\beta+k+l,\beta+k-l)}$ which vanishes for $k\ge n+1$, the resulting expression vanishes for all these k values which results in a terminating sum over $k\in\{0,\ldots,n\}$.

Unlike Koornwinder's addition theorem for Jacobi polynomials, the addition theorem for the Jacobi functions of the second kind (see §4) is not a terminating sum. One can see this by examination of (151), by recognizing that the Jacobi polynomials $P_{\gamma-k}^{(\alpha+k+l,\beta+k-l)}$ vanish for $\gamma=n\in\mathbb{N}_0$ and $k\ge n+1$. However, considering the limit as $\gamma\to n$ for all values of $k\in\mathbb{N}_0$ in Koornwinder's addition theorem, the factor $1/(-\gamma)_k$ blows up for $k\ge n+1$. On the other hand, this factor in the limit is multiplied by the Jacobi function of the first kind prefactor containing $1/\Gamma(\gamma-k+1)$, while considering the residues of the gamma function, the product will be finite, namely (153), for $k\ge n+1$. This finite factor is then multiplied by the Jacobi function of the second kind $Q_{\gamma-k}^{(\alpha+\beta+k+l,\beta+k-l)}(2z_>^2-1)$ which does not vanish for $k\ge n+1$ for $\alpha,\beta\notin\mathbb{Z}$, unlike the case for Jacobi polynomials.

4.4 Olver normalized Jacobi functions

We previously introduced Olver's normalization of the Gauss [48, (15.2.2)] and generalized hypergeometric function (14) (see also [48, (16.2.5)]) which results in these functions being entire functions of all of the parameters which appear including all denominator factors. Olver applied this concept of special normalization previously to the associated Legendre function of the second kind [48, (14.3.10)] (see also [45, p. 170 and 178]). We now demonstrate how to apply this concept to the Jacobi functions of the first and second kind.

In the above description, instead of carefully determining the limits of the relevant functions when there are removable singularities due to the appearance of various gamma function prefactors, an alternative option is to use appropriately defined Olver normalized Jacobi functions and recast the addition theorems correspondingly. The benefit of using Olver normalized definitions of the Jacobi functions is that one avoids complications due to gamma functions with removable singularities. Typical examples of these benefits occur in the often appearing examples when one has degrees γ and parameters α,β given by integers. In these cases, using the standard definitions such as those which appear in Theorems 5, 8, the appearing functions are not defined and careful limits must be taken. However, if one adopts carefully chosen Olver normalized definitions where only the Olver normalized Gauss hypergeometric functions are used, then these functions will be entire for all values of the parameters. As we will see, by using these definitions, we arrive at formulas for the addition theorems which are elegant and highly useful! First we give our new choice of the Olver normalization and then give the relations of the Olver normalized definitions in terms of the usual definitions. Our definitions of Olver normalized Jacobi functions of the first and second kind in the hyperbolic and trigonometric contexts are given by

$$\boldsymbol{P}_\gamma^{(\alpha,\beta)}(z) := {}_2\boldsymbol{F}_1\left(\begin{matrix}-\gamma, \alpha+\beta+\gamma+1\\ \alpha+1\end{matrix}; \frac{1-z}{2}\right), \tag{154}$$

$$\boldsymbol{Q}_\gamma^{(\alpha,\beta)}(z) := \frac{2^{\alpha+\beta+\gamma}}{(z-1)^{\alpha+\gamma+1}(z+1)^\beta}\, {}_2\boldsymbol{F}_1\left(\begin{matrix}\gamma+1, \alpha+\gamma+1\\ \alpha+\beta+2\gamma+2\end{matrix}; \frac{2}{1-z}\right), \tag{155}$$

$$\mathsf{P}_\gamma^{(\alpha,\beta)}(x) := {}_2\boldsymbol{F}_1\left(\begin{matrix}-\gamma, \alpha+\beta+\gamma+1\\ \alpha+1\end{matrix}; \frac{1-x}{2}\right), \tag{156}$$

$$\begin{aligned}\mathsf{Q}_\gamma^{(\alpha,\beta)}(x) := \tfrac{1}{2}\Gamma(\alpha+1)\left(\frac{1+x}{2}\right)^\gamma \left(\frac{\cos(\pi\alpha)\Gamma(\alpha+\gamma+1)}{\Gamma(\gamma+1)}\, {}_2\boldsymbol{F}_1\left(\begin{matrix}-\gamma, -\beta-\gamma\\ 1+\alpha\end{matrix}; \frac{x-1}{x+1}\right)\right.\\ \left. -\frac{\Gamma(\beta+\gamma+1)}{\Gamma(\alpha+\beta+\gamma+1)}\left(\frac{1+x}{1-x}\right)^\alpha {}_2\boldsymbol{F}_1\left(\begin{matrix}-\alpha-\gamma, -\alpha-\beta-\gamma\\ 1-\alpha\end{matrix}; \frac{x-1}{x+1}\right)\right).\end{aligned} \tag{157}$$

Therefore one has the following connection relations between the Jacobi functions of the first and second kinds and their Olver normalized counterparts, namely

$$P_\gamma^{(\alpha,\beta)}(z) = \frac{\Gamma(\alpha+\gamma+1)}{\Gamma(\gamma+1)}\boldsymbol{P}_\gamma^{(\alpha,\beta)}(z), \tag{158}$$

$$Q_\gamma^{(\alpha,\beta)}(z) = \Gamma(\alpha+\gamma+1)\Gamma(\beta+\gamma+1)\, \boldsymbol{Q}_\gamma^{(\alpha,\beta)}(z), \tag{159}$$

$$\mathsf{P}_\gamma^{(\alpha,\beta)}(x) = \frac{\Gamma(\alpha+\gamma+1)}{\Gamma(\gamma+1)}\boldsymbol{\mathsf{P}}_\gamma^{(\alpha,\beta)}(x). \tag{160}$$

Note that

$$\boldsymbol{\mathsf{P}}_\gamma^{(\alpha,\beta)}(x) = \boldsymbol{P}_\gamma^{(\alpha,\beta)}(x \pm i0), \tag{161}$$

as in (73). Furthermore in the special case $\gamma = 0$ one has

$$\boldsymbol{Q}_0^{(\alpha,\beta)}(z) := \frac{2^{\alpha+\beta}}{(z-1)^{\alpha+1}(z+1)^\beta}\, {}_2\boldsymbol{F}_1\left(\begin{matrix}1, \alpha+1\\ \alpha+\beta++2\end{matrix}; \frac{2}{1-z}\right). \tag{162}$$

Remark 50. *As of the date of publication of this manuscript, we have been unable to find an Olver normalized version of the Jacobi function of the second kind on-the-cut* $\boldsymbol{\mathsf{Q}}_\gamma^{(\alpha,\beta)}(x)$*. However we did find a special normalization of this function which works well when* $\gamma = 0$ *and the* β *parameters take integer values which is of particular importance because they appear in a very important application (see Section 5 below). Let* $b \in \mathbb{N}_0$*. Define*

$$\mathcal{Q}_{-k}^{(\alpha+k+l,b+k-l)}(x) := \lim_{\gamma\to 0,\beta\to b}(-\beta-\gamma)_l \mathsf{Q}_{\gamma-k}^{(\alpha+k+l,\beta+k-l)}(x), \tag{163}$$

which is a well-defined function for all α, b, x, k, l *in its domain.*

4.5 Addition theorems for the Olver normalized Jacobi functions

Now that we've introduced the Olver normalized Jacobi functions of the first and second kind in the hyperbolic and trigonometric contexts, we are in a position to perform the straightforward derivation of the corresponding addition theorems for these functions.

Theorem 51. *Let* $\gamma,\alpha,\beta\in\mathbb{C}$, $z_1,z_2\in\mathbb{C}\setminus(-\infty,1]$, $x_1,x_2\in\mathbb{C}\setminus((-\infty,-1]\cup[1,\infty))$, $x,w\in\mathbb{C}$, *and* $Z^\pm$, $\mathsf{X}^\pm$ *as defined in* (105), (106), *respectively, such that the complex variables* $\gamma,\alpha,\beta,z_1,z_2,x_1,x_2,x,w$ *are in some yet to be determined neighborhood of the real line. Then*

$$P_\gamma^{(\alpha,\beta)}(Z^\pm)=\frac{\Gamma(\alpha+1)\Gamma(\alpha+\gamma+1)}{\Gamma(\gamma+1)}\sum_{k=0}^{\infty}\frac{(\alpha+1)_k(\alpha+\beta+\gamma+1)_k(-\gamma)_k}{(\alpha+k)(\beta+1)_k}$$
$$\times\sum_{l=0}^{k}(\mp1)^{k-l}(\alpha+k+l)(\alpha+\gamma+1)_l(-\beta-\gamma)_l(z_1z_2)^{k-l}((z_1^2-1)(z_2^2-1))^{\frac{k+l}{2}}$$
$$\times\boldsymbol{P}_{\gamma-k}^{(\alpha+k+l,\beta+k-l)}(2z_1^2-1)\boldsymbol{P}_{\gamma-k}^{(\alpha+k+l,\beta+k-l)}(2z_2^2-1)w^{k-l}P_l^{(\alpha-\beta-1,\beta+k-l)}(2w^2-1)\frac{\beta+k-l}{\beta}C_{k-l}^{\beta}(x),\quad(164)$$

$$Q_\gamma^{(\alpha,\beta)}(Z^\pm)=\Gamma(\alpha+1)\Gamma(\alpha+\gamma+1)\Gamma(\beta+\gamma+1)\sum_{k=0}^{\infty}\frac{(\alpha+1)_k(\alpha+\beta+\gamma+1)_k}{(\alpha+k)(\beta+1)_k}$$
$$\times\sum_{l=0}^{k}(\mp1)^{k-l}(\alpha+k+l)(\alpha+\gamma+1)_l(z_1z_2)^{k-l}((z_1^2-1)(z_2^2-1))^{\frac{k+l}{2}}$$
$$\times\boldsymbol{Q}_{\gamma-k}^{(\alpha+k+l,\beta+k-l)}(2z_>^2-1)\boldsymbol{P}_{\gamma-k}^{(\alpha+k+l,\beta+k-l)}(2z_<^2-1)w^{k-l}P_l^{(\alpha-\beta-1,\beta+k-l)}(2w^2-1)\frac{\beta+k-l}{\beta}C_{k-l}^{\beta}(x),\quad(165)$$

$$\mathsf{P}_\gamma^{(\alpha,\beta)}(\mathsf{X}^\pm)=\frac{\Gamma(\alpha+1)\Gamma(\alpha+\gamma+1)}{\Gamma(\gamma+1)}\sum_{k=0}^{\infty}\frac{(\alpha+1)_k(\alpha+\beta+\gamma+1)_k(-\gamma)_k}{(\alpha+k)(\beta+1)_k}$$
$$\times\sum_{l=0}^{k}(\mp1)^{k-l}(\alpha+k+l)(\alpha+\gamma+1)_l(-\beta-\gamma)_l(x_1x_2)^{k-l}((1-x_1^2)(1-x_2^2))^{\frac{k+l}{2}}$$
$$\times\boldsymbol{\mathsf{P}}_{\gamma-k}^{(\alpha+k+l,\beta+k-l)}(2x_1^2-1)\boldsymbol{\mathsf{P}}_{\gamma-k}^{(\alpha+k+l,\beta+k-l)}(2x_2^2-1)w^{k-l}P_l^{(\alpha-\beta-1,\beta+k-l)}(2w^2-1)\frac{\beta+k-l}{\beta}C_{k-l}^{\beta}(x),\quad(166)$$

$$\mathsf{Q}_\gamma^{(\alpha,\beta)}(\mathsf{X}^\pm)=\Gamma(\alpha+1)\sum_{k=0}^{\infty}\frac{(-1)^k(\alpha+\beta+\gamma+1)_k(\alpha+1)_k}{(\alpha+k)(\beta+1)_k}$$
$$\times\sum_{l=0}^{k}(\mp1)^{k-l}(\alpha+k+l)(-\beta-\gamma)_l(x_1x_2)^{k-l}((1-x_1^2)(1-x_2^2))^{\frac{k+l}{2}}$$
$$\times\mathsf{Q}_{\gamma-k}^{(\alpha+k+l,\beta+k-l)}(2x_<^2-1)\boldsymbol{\mathsf{P}}_{\gamma-k}^{(\alpha+k+l,\beta+k-l)}(2x_>^2-1)w^{k-l}P_l^{(\alpha-\beta-1,\beta+k-l)}(2w^2-1)\frac{\beta+k-l}{\beta}C_{k-l}^{\beta}(x).\quad(167)$$

Proof. Substituting (158)–(160) into (107), (108), (133), (151) as necessary completes the proof. ■

There are also the corresponding expansions that are sometimes useful with the l sum reversed, i.e., making the replacement $l'=k-l$ and then replacing $l'\mapsto l$ in Theorem 39. These are given as follows.

Corollary 52. *Let* $\gamma,\alpha,\beta\in\mathbb{C}$, $z_1,z_2\in\mathbb{C}\setminus(-\infty,1]$, $x_1,x_2\in\mathbb{C}\setminus((-\infty,-1]\cup[1,\infty))$, $x,w\in\mathbb{C}$, *with* $Z^\pm$, $\mathsf{X}^\pm$ *as defined in* (105), (106), *respectively, and the complex variables* $\gamma,\alpha,\beta,z_1,z_2,x_1,x_2,x,w$ *are in some yet to* *be determined neighborhood of the real line. Then*

$$P_\gamma^{(\alpha,\beta)}(Z^\pm)=\frac{\Gamma(\alpha+1)\Gamma(\alpha+\gamma+1)}{\Gamma(\gamma+1)}\sum_{k=0}^\infty\frac{(\alpha+1)_k(\alpha+\beta+\gamma+1)_k(-\gamma)_k}{(\alpha+k)(\beta+1)_k}$$
$$\times\sum_{l=0}^k(\mp1)^l(\alpha+2k-l)(\alpha+\gamma+1)_{k-l}(-\beta-\gamma)_{k-l}(z_1z_2)^l((z_1^2-1)(z_2^2-1))^{\frac{2k-l}{2}}$$
$$\times\boldsymbol{P}_{\gamma-k}^{(\alpha+2k-l,\beta+l)}(2z_1^2-1)\boldsymbol{P}_{\gamma-k}^{(\alpha+2k-l,\beta+l)}(2z_2^2-1)w^lP_{k-l}^{(\alpha-\beta-1,\beta+l)}(2w^2-1)\frac{\beta+l}{\beta}C_l^\beta(x), \tag{168}$$

$$Q_\gamma^{(\alpha,\beta)}(Z^\pm)=\Gamma(\alpha+1)\Gamma(\alpha+\gamma+1)\Gamma(\beta+\gamma+1)\sum_{k=0}^\infty\frac{(\alpha+1)_k(\alpha+\beta+\gamma+1)_k}{(\alpha+k)(\beta+1)_k}$$
$$\times\sum_{l=0}^k(\mp1)^l(\alpha+2k-l)(\alpha+\gamma+1)_{k-l}(z_1z_2)^l((z_1^2-1)(z_2^2-1))^{\frac{2k-l}{2}}$$
$$\times\boldsymbol{Q}_{\gamma-k}^{(\alpha+2k-l,\beta+l)}(2z_>^2-1)\boldsymbol{P}_{\gamma-k}^{(\alpha+2k-l,\beta+l)}(2z_<^2-1)w^lP_{l}^{(\alpha-\beta-1,\beta+l)}(2w^2-1)\frac{\beta+l}{\beta}C_l^\beta(x), \tag{169}$$

$$\mathsf{P}_\gamma^{(\alpha,\beta)}(\mathsf{X}^\pm)=\frac{\Gamma(\alpha+1)\Gamma(\alpha+\gamma+1)}{\Gamma(\gamma+1)}\sum_{k=0}^\infty\frac{(\alpha+1)_k(\alpha+\beta+\gamma+1)_k(-\gamma)_k}{(\alpha+k)(\beta+1)_k}$$
$$\times\sum_{l=0}^k(\mp1)^l(\alpha+2k-l)(\alpha+\gamma+1)_{k-l}(-\beta-\gamma)_{k-l}(x_1x_2)^l((x_1^2-1)(x_2^2-1))^{\frac{2k-l}{2}}$$
$$\times\boldsymbol{\mathsf{P}}_{\gamma-k}^{(\alpha+2k-l,\beta+l)}(2x_1^2-1)\boldsymbol{\mathsf{P}}_{\gamma-k}^{(\alpha+2k-l,\beta+l)}(2x_2^2-1)w^lP_{k-l}^{(\alpha-\beta-1,\beta+l)}(2w^2-1)\frac{\beta+l}{\beta}C_l^\beta(x), \tag{170}$$

$$\mathsf{Q}_\gamma^{(\alpha,\beta)}(\mathsf{X}^\pm)=\Gamma(\alpha+1)\sum_{k=0}^\infty(-1)^k\frac{(\alpha+1)_k(\alpha+\beta+\gamma+1)_k}{(\alpha+k)(\beta+1)_k}$$
$$\times\sum_{l=0}^k(\mp1)^l(\alpha+2k-l)(-\beta-\gamma)_{k-l}(x_1x_2)^l((x_1^2-1)(x_2^2-1))^{\frac{2k-l}{2}}$$
$$\times\mathsf{Q}_{\gamma-k}^{(\alpha+2k-l,\beta+l)}(2x_<^2-1)\boldsymbol{\mathsf{P}}_{\gamma-k}^{(\alpha+2k-l,\beta+l)}(2x_>^2-1)w^lP_{k-l}^{(\alpha-\beta-1,\beta+l)}(2w^2-1)\frac{\beta+l}{\beta}C_l^\beta(x). \tag{171}$$

Proof. Making the replacement $l\mapsto k-l$ in Theorem 51 completes the proof. ■

Remark 53. *It should be noted that moving the factors* $\Gamma(\alpha+\gamma+1)/\Gamma(\gamma+1)$ *for the* P *series and* $\Gamma(\alpha+\gamma+1)\Gamma(\beta+\gamma+1)$ *for the* Q *series converts the* P *and* Q *on the left to* $\boldsymbol{P}$ *and* $\boldsymbol{Q}$ *so the whole expressions are in terms of the Olver normalized functions.*

By examining the expansion of the Jacobi function of the first kind, one can see that the $(-\gamma)_k$ shifted factorial in this alternative expansion is moved from the denominator to the numerator, so it is more natural for Jacobi polynomials where the sum is terminating. One can see the benefit is that all the functions involved in the expansions are well-defined for all values of the parameters, including integer values. These expansions are extremely useful for expansions of fundamental solutions of rank-one symmetric spaces where all the degrees and parameters are given by integers. One no longer has any difficulties with the various functions not being defined for certain parameter values. This is completely resolved. One example is that in the integer context for the Jacobi function of the second kind, the functions appear with degree equal to $\gamma-k$ for all $k\in\mathbb{N}_0$. These quickly become undefined for negative values of the degree. However, since the Olver normalized Jacobi functions are entire functions, there is no longer any problem here. These alternative expansions are highly desirable!

5 Eigenfunction expansions of a fundamental solution of the Laplace-Beltrami operator on non-compact and compact symmetric spaces of rank one

As an application of the addition theorem for the Jacobi function of the second kind, we now give an introduction to the motivation of the material which has thus far been presented in the previous sections. It is a study of the global analysis of the Laplace-Beltrami operator and solutions to inhomogeneous elliptic equation given by Poisson's equation on the Riemannian symmetric spaces of rank one.

Let $d = \dim_{\mathbb{R}} \mathbb{K}$, where $\mathbb{K}$ is equal to either the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$, the quaternions $\mathbb{H}$ or the octonions $\mathbb{O}$. For $d = 1$, namely the real case, these are Riemannian manifolds of constant curvature which include Euclidean $\mathbb{R}^n$ space, real hyperbolic geometry $\mathbb{R}\mathbf{H}^n_R$ (noncompact) and real hyperspherical (compact) geometry $\mathbb{R}\mathbf{S}^n_R$, in various models. For $d \in \{2, 4, 8\}$, it is well known that there exists isotropic Riemannian manifolds of both noncompact and compact type which are referred to as the rank one symmetric spaces, see for instance [27]. These include the symmetric spaces given by the complex hyperbolic $\mathbb{C}\mathbf{H}^n_R$, quaternionic hyperbolic $\mathbb{H}\mathbf{H}^n_R$, and the octonionic hyperbolic plane $\mathbb{O}\mathbf{H}^2_R$ and the complex projective space $\mathbb{C}\mathbf{P}^n_R$, quaternionic projective space $\mathbb{H}\mathbf{P}^n_R$, and the octonionic projective (Cayley) plane $\mathbb{O}\mathbf{P}^2_R$, where $R > 0$ is their corresponding radii of curvatures. The complex, quaternionic and octonionic rank one symmetric spaces have real dimension given by $2n$, $4n$ and 16. For a description of the Riemannian manifolds given by the rank one symmetric spaces, see for instance [25, 26, 27] and the references therein.

Riemannian symmetric spaces, compact and non-compact, come in infinite series (4 corresponding to simple complex groups and 7 corresponding to real simple groups) and a finite class of exceptional spaces, see [25, p. 516, 518]. Each of those come with a commutative algebra of invariant differential operators and correspondingly, a class of eigenfunctions, the spherical functions. In the case of the rank-one symmetric spaces, these are just hypergeometric (Jacobi) functions with specified parameters. This has been used as motivation for several generalizations of the classical Gauss hypergeometric functions that are beyond the scope of this paper, for example the Heckman-Opdam functions on root systems [22, 23, 46, 47] and the work of Macdonald [41, 42].

Due to the isotropy of the symmetric spaces of rank one, a fundamental solution of the Laplace-Beltrami operator on these manifolds can be obtained by solving a one-dimensional ordinary differential equation given in terms of the geodesic distance. Laplace's equation is satisfied on these manifolds when the Laplace-Beltrami operator acts on an unknown function and the result is zero. In geodesic polar coordinates the Laplace-Beltrami operator is given in the rank one noncompact (hyperbolic) symmetric spaces by

$$\Delta = \frac{1}{R^2}\left\{\frac{\partial^2}{\partial r^2} + [d(n-1)\coth r + 2(d-1)\coth(2r)]\frac{\partial}{\partial r} + \frac{1}{\sinh^2 r}\Delta_{K/M}\right\} \tag{172}$$

$$= \frac{1}{R^2}\left\{\frac{\partial^2}{\partial r^2} + [(dn-1)\coth r + (d-1)\tanh r]\frac{\partial}{\partial r} + \frac{1}{\sinh^2 r}\Delta_{K/M}\right\} \tag{173}$$

$$=: \frac{1}{R^2}\left(\Delta_r + \frac{1}{\sinh^2 r}\Delta_{K/M}\right), \tag{174}$$

and on the rank one compact (projective) spaces it is given by

$$\Delta = \frac{1}{R^2}\left\{\frac{\partial^2}{\partial \theta^2} + [d(n-1)\cot\theta + 2(d-1)\cot(2\theta)]\frac{\partial}{\partial \theta} + \frac{1}{\sin^2\theta}\Delta_{K/M}\right\} \tag{175}$$

$$= \frac{1}{R^2}\left\{\frac{\partial^2}{\partial \theta^2} + [(dn-1)\cot\theta + (d-1)\tan\theta]\frac{\partial}{\partial \theta} + \frac{1}{\sin^2\theta}\Delta_{K/M}\right\} \tag{176}$$

$$=: \frac{1}{R^2}\left(\Delta_\theta + \frac{1}{\sin^2\theta}\Delta_{K/M}\right), \tag{177}$$

where r and θ are the geodesic distance on the noncompact and compact rank one symmetric spaces respectively (see [25, Lemma 21]). For a spherically symmetric solution such as a fundamental solution, the contribution from $\Delta_{K/M}$ vanishes and one needs to solve Laplace's equation for radial solutions, namely

$$\Delta_r\, u(r) = 0, \quad \Delta_\theta\, v(\theta) = 0. \tag{178}$$

For the solution to these equations, the homogeneous solutions to the second order ordinary differential equation which appears are given by Jacobi/hypergeometric functions (see [27, p. 484]). It can be easily verified that a basis for radial solutions can be given by

$$u(r) = a\, P_0^{(\alpha,\beta)}(\cosh(2r)) + b\, Q_0^{(\alpha,\beta)}(\cosh(2r)). \tag{179}$$

$$v(\theta) = c\, \mathsf{P}_0^{(\alpha,\beta)}(\cos(2\theta)) + d\, \mathsf{Q}_0^{(\alpha,\beta)}(\cos(2\theta)), \tag{180}$$

where on the complex, quaternionic and octonionic rank one symmetric spaces one has $\alpha \in \{n-1, 2n-1, 7\}$ and $\beta \in \{0, 1, 3\}$ respectively [18, Table 1, p. 265]. Furthermore, for a fundamental solution, the solutions need to be singular at the origin and match up to a Euclidean fundamental solutions locally. This requires that the solutions should be irregular at the origin ($r = 0$ and $\theta = 0$). Therefore fundamental solutions must correspond to the solutions which are the functions of the second kind. Hence for a fundamental solution of Laplace's equation $a = c = 0$ and we must determine b and d which will be a function of d, n and R.

Remark 54. *Note that the general homogeneous solution as a function of the geodesic coordinate includes contribution from both the function of the first kind and function of the second kind. However, the function of the first kind with $\gamma = 0$ is simply the constants a and c since $P_0^{(\alpha,\beta)}(z) = 1$* (58) *(same for the functions on-the-cut). On the other hand, in the case of non-spherically symmetric homogeneous solutions there will be contributions due to the function of the first kind because then the contribution to the $\Delta_{K/M}$ term will be non-zero.*

Let $\mathbf{x}, \mathbf{x}' \in \mathbb{R}^s$, then a Euclidean fundamental solution of Laplace's equation is given by (see for instance [21, p. 202])

$$\mathcal{G}^s(\mathbf{x}, \mathbf{x}') = \begin{cases} \dfrac{\Gamma(s/2)}{2\pi^{s/2}(s-2)} \|\mathbf{x} - \mathbf{x}'\|^{2-s} & \text{if } s = 1 \text{ or } s \geq 3, \\ \dfrac{1}{2\pi} \log \|\mathbf{x} - \mathbf{x}'\|^{-1} & \text{if } s = 2. \end{cases} \tag{181}$$

For a description of opposite antipodal fundamental solutions on the real hypersphere see [8]. The above analysis leads us to the following theorem.

Theorem 55. *A fundamental solution and an opposite antipodal fundamental solution of the Laplace-Beltrami operator on the rank one noncompact and compact symmetric spaces respectively, $n \geq 1$, given in terms of the geodesic radii $r \in [0, \infty)$, $\theta \in [0, \pi/2]$ on these manifolds are given by*

$$\mathcal{G}^{\mathbb{C}\mathbf{H}_R^n}(r) = \frac{(n-1)!}{2\pi^n R^{2n-2}} Q_0^{(n-1,0)}(\cosh(2r)), \tag{182}$$

$$\mathcal{G}^{\mathbb{H}\mathbf{H}_R^n}(r) = \frac{(2n)!}{2\pi^{2n} R^{4n-2}} Q_0^{(2n-1,1)}(\cosh(2r)), \tag{183}$$

$$\mathcal{G}^{\mathbb{O}\mathbf{H}_R^2}(r) = \frac{302\,400}{\pi^8 R^{14}} Q_0^{(7,3)}(\cosh(2r)), \tag{184}$$

$$\mathcal{G}^{\mathbb{C}\mathbf{P}_R^n}(\theta) = \frac{(n-1)!}{2\pi^n R^{2n-2}} \mathsf{Q}_0^{(n-1,0)}(\cos(2\theta)), \tag{185}$$

$$\mathcal{G}^{\mathbb{H}\mathbf{P}_R^n}(\theta) = \frac{(2n)!}{2\pi^{2n} R^{4n-2}} \mathsf{Q}_0^{(2n-1,1)}(\cos(2\theta)), \tag{186}$$

$$\mathcal{G}^{\mathbb{O}\mathbf{P}_R^2}(\theta) = \frac{302\,400}{\pi^8 R^{14}} \mathsf{Q}_0^{(7,3)}(\cos(2\theta)). \tag{187}$$

Proof. The complex, quaternionic and octonionic rank one symmetric spaces all have even dimensions, namely $s \in \{2n, 4n, 16\}$, respectively. It is easy to verify that the homogeneous spherically symmetric solutions of Laplace's equation on the complex, quaternionic and octonionic rank one symmetric spaces are given by Jacobi functions of the first and second kind for the noncompact manifolds and are given by Jacobi functions of the first and second kind on-the-cut for the compact manifolds, both having $\gamma = 0$, $\alpha \in \{n-1, 2n-1, 7\}$ and $\beta \in \{0, 1, 3\}$ respectively. Furthermore, one requires that locally these fundamental solutions match up to a Euclidean fundamental solution. Using (71), (79), assuming $\gamma = 0$, $\alpha = a$, $\beta = b$, $a \in \mathbb{N}$, $b \in \mathbb{N}_0$, one has the following behaviors near the singularity at unity for the Jacobi function of the second kind and the Jacobi function of the second kind on-the-cut, for $\epsilon \to 0^+$,

$$Q_0^{(a,b)}(1+\epsilon) \sim \mathsf{Q}_0^{(a,b)}(1-\epsilon) \sim \frac{2^{a-1}(a-1)!b!}{(a+b)!\,\epsilon^a}. \tag{188}$$

Referring to the geodesic distance on the hyperbolic manifolds as $r \in [0, \infty)$ and on the compact manifolds as $\theta \in [0, \pi/2]$, one has

$$\cosh(2r) \sim \cosh(2\tfrac{\rho}{R}) \sim 1 + \tfrac{2\rho^2}{R^2}, \tag{189}$$

$$\cos(2\theta) \sim \cos(\tfrac{2\rho}{R}) \sim 1 - \tfrac{2\rho^2}{R^2}, \tag{190}$$

where ρ is the Euclidean geodesic distance. Matching locally to a Euclidean fundamental solution (181) using the flat-space limit (see for instance [10, §2.4]), one is able to determine the constants of proportionality which are multiplied by the Jacobi functions of the second kind. This completes the proof. ∎

Since fundamental solutions on the rank one symmetric spaces all have $\gamma = 0$, we first present the expansions in these cases. For the Jacobi functions of the first kind the $\gamma = 0$ case just corresponds with unity. However, for the Jacobi functions of the second kind, these functions are quite rich, and the expansions are quite useful in that they allow one to produce separated eigenfunction expansions of a fundamental solution of Laplace's equation on these isotropic spaces.

Remark 56. *The reader should be aware that the addition theorems presented below for the Jacobi functions of the second kind with $\gamma = 0$ are well-defined except in the case where the α and β parameters on the left-hand sides are non-negative integers. In that case, special care must be taken (refer to Theorems 12, 13), even though the functions at these parameter values may be obtained by taking the appropriate limit.*

Corollary 57. *Let $\alpha, \beta \in \mathbb{C}$, $\beta \notin \mathbb{Z}$, $z_1, z_2 \in \mathbb{C} \setminus (-\infty, 1]$, $x_1, x_2 \in \mathbb{C} \setminus ((-\infty, -1] \cup [1, \infty))$, $x, w \in \mathbb{C}$, with $Z^\pm$, $\mathsf{X}^\pm$ as defined in* (105), (106), *respectively, such that the complex variables $\alpha, \beta, z_1, z_2, x_1, x_2, x, w$ are in some yet to be determined neighborhood of the real line. Then*

$$\begin{aligned} Q_0^{(\alpha,\beta)}(Z^\pm) &= \Gamma(\alpha+1)\Gamma(\alpha+1)\Gamma(\beta+1)\sum_{k=0}^{\infty}\frac{(\alpha+1)_k(\alpha+\beta+1)_k}{(\alpha+k)(\beta+1)_k} \\ &\times\sum_{l=0}^{k}(\mp1)^{k-l}(\alpha+k+l)(\alpha+1)_l(z_1z_2)^{k-l}((z_1^2-1)(z_2^2-1))^{\frac{k+l}{2}} \\ &\times\boldsymbol{Q}_{-k}^{(\alpha+k+l,\beta+k-l)}(2z_>^2-1)\boldsymbol{P}_{-k}^{(\alpha+k+l,\beta+k-l)}(2z_<^2-1)w^{k-l}P_l^{(\alpha-\beta-1,\beta+k-l)}(2w^2-1)\frac{\beta+k-l}{\beta}C_{k-l}^{\beta}(x), \end{aligned} \tag{191}$$

$$\begin{aligned} \mathsf{Q}_0^{(\alpha,\beta)}(\mathsf{X}^\pm) &= \Gamma(\alpha+1)\sum_{k=0}^{\infty}(-1)^k\frac{(\alpha+1)_k(\alpha+\beta+1)_k}{(\alpha+k)(\beta+1)_k} \\ &\times\sum_{l=0}^{k}(\mp1)^{k-l}(\alpha+k+l)(-\beta)_l(x_1x_2)^{k-l}((1-x_1^2)(1-x_2^2))^{\frac{k+l}{2}} \\ &\times\mathsf{Q}_{-k}^{(\alpha+k+l,\beta+k-l)}(2x_<^2-1)\mathsf{P}_{-k}^{(\alpha+k+l,\beta+k-l)}(2x_>^2-1)w^{k-l}P_l^{(\alpha-\beta-1,\beta+k-l)}(2w^2-1)\frac{\beta+k-l}{\beta}C_{k-l}^{\beta}(x). \end{aligned} \tag{192}$$

Proof. Substituting $\gamma = 0$ in Theorem 51 for the Jacobi functions of the second kind completes the proof. ■

Next we give examples of the expansions for complex and quaternionic hyperbolic spaces where $\beta \in \{0, 1, 3\}$ respectively. First we treat the complex case which corresponds to complex hyperbolic space and complex projective space. In order to do this we start with Corollary 51 and take the limit as $\beta \to 0$ using [3, (6.4.13)]

$$\lim_{\mu\to 0} \frac{n+\mu}{\mu} C_n^\mu(x) = \epsilon_n T_n(x), \tag{193}$$

where $\epsilon_n := 2 - \delta_{n,0}$ is the Neumann factor commonly appearing in Fourier series.

Corollary 58. *Let $\alpha \in \mathbb{C}$, $z_1, z_2 \in \mathbb{C} \setminus (-\infty, 1]$, $x_1, x_2 \in \mathbb{C} \setminus ((-\infty, -1] \cup [1, \infty))$, $x, w \in \mathbb{C}$, with $Z^\pm$, $\mathsf{X}^\pm$ as defined in* (105), (106)*, respectively, such that the complex variables $\alpha, z_1, z_2, x_1, x_2, x, w$ are in some yet to be determined neighborhood of the real line. Then*

$$\begin{aligned} Q_0^{(\alpha,0)}(Z^\pm) &= \Gamma(\alpha+1)\Gamma(\alpha+1) \sum_{k=0}^\infty \frac{(\alpha+1)_k(\alpha+1)_k}{(\alpha+k)k!} \\ &\quad\times \sum_{l=0}^k (\mp 1)^{k-l} (\alpha+k+l)(\alpha+1)_l (z_1z_2)^{k-l} ((z_1^2-1)(z_2^2-1))^{\frac{k+l}{2}} \\ &\qquad \times \boldsymbol{Q}_{-k}^{(\alpha+k+l,k-l)}(2z_>^2-1) \boldsymbol{P}_{-k}^{(\alpha+k+l,k-l)}(2z_<^2-1) w^{k-l} P_l^{(\alpha-1,k-l)}(2w^2-1)\epsilon_{k-l}T_{k-l}(x), \end{aligned} \tag{194}$$

$$\begin{aligned} \mathsf{Q}_0^{(\alpha,0)}(\mathsf{X}^\pm) &= \Gamma(\alpha+1) \sum_{k=0}^\infty (-1)^k \frac{(\alpha+1)_k(\alpha+1)_k}{(\alpha+k)k!} \\ &\quad\times \sum_{l=0}^k (\mp 1)^{k-l} (\alpha+k+l)(x_1x_2)^{k-l} ((1-x_1^2)(1-x_2^2))^{\frac{k+l}{2}} \\ &\qquad \times \mathcal{Q}_{-k}^{(\alpha+k+l,k-l)}(2x_<^2-1) \mathsf{P}_{-k}^{(\alpha+k+l,k-l)}(2x_>^2-1) w^{k-l} P_l^{(\alpha-1,k-l)}(2w^2-1)\epsilon_{k-l}T_{k-l}(x). \end{aligned} \tag{195}$$

Proof. Take the limit as $\beta \to 0$ in Corollary 51 using (193) completes the proof. ■

Now we treat the case corresponding to the quaternionic hyperbolic and projective spaces which correspond to $\beta = 1$.

Corollary 59. *Let $\alpha \in \mathbb{C}$, $z_1, z_2 \in \mathbb{C} \setminus (-\infty, 1]$, $x_1, x_2 \in \mathbb{C} \setminus ((-\infty, -1] \cup [1, \infty))$, $x, w \in \mathbb{C}$, with $Z^\pm$, $\mathsf{X}^\pm$ as defined in* (105), (106)*, respectively, such that the complex variables $\alpha, z_1, z_2, x_1, x_2, x, w$ are in some yet to be determined neighborhood of the real line. Then*

$$\begin{aligned} Q_0^{(\alpha,1)}(Z^\pm) &= \Gamma(\alpha+1)\Gamma(\alpha+1) \sum_{k=0}^\infty \frac{(\alpha+1)_k(\alpha+2)_k}{(\alpha+k)(2)_k} \\ &\quad\times \sum_{l=0}^k (\mp 1)^{k-l} (1+k-l)(\alpha+k+l)(\alpha+1)_l (z_1z_2)^{k-l} ((z_1^2-1)(z_2^2-1))^{\frac{k+l}{2}} \\ &\qquad \times \boldsymbol{Q}_{-k}^{(\alpha+k+l,1+k-l)}(2z_>^2-1) \boldsymbol{P}_{-k}^{(\alpha+k+l,1+k-l)}(2z_<^2-1) w^{k-l} P_l^{(\alpha-2,1+k-l)}(2w^2-1)U_{k-l}(x), \end{aligned} \tag{196}$$

$$\begin{aligned} \mathsf{Q}_0^{(\alpha,1)}(\mathsf{X}^\pm) &= \Gamma(\alpha+1) \sum_{k=0}^\infty (-1)^k \frac{(\alpha+1)_k(\alpha+2)_k}{(\alpha+k)(2)_k} \\ &\quad\times \sum_{l=0}^k (\mp 1)^{k-l} (1+k-l)(\alpha+k+l)(x_1x_2)^{k-l} ((1-x_1^2)(1-x_2^2))^{\frac{k+l}{2}} \\ &\qquad \times \mathcal{Q}_{-k}^{(\alpha+k+l,1+k-l)}(2x_<^2-1) \mathsf{P}_{-k}^{(\alpha+k+l,1+k-l)}(2x_>^2-1) w^{k-l} P_l^{(\alpha-2,1+k-l)}(2w^2-1)U_{k-l}(x). \end{aligned} \tag{197}$$

Proof. Take the limit as $\beta \to 1$ in Corollary 51 using [48, (18.7.4)] which connects the Chebyshev polynomial of the second kind to the Gegenbauer polynomial with parameter equal to unity, namely $C_n^1(x) = U_n(x)$. This completes the proof. ■

Now we treat the case corresponding to the octonionic hyperbolic space and octonionic projective space. This corresponds to $\beta = 3$.

Corollary 60. *Let $\alpha \in \mathbb{C}$, $z_1, z_2 \in \mathbb{C} \setminus (-\infty, 1]$, $x_1, x_2 \in \mathbb{C} \setminus ((-\infty, -1] \cup [1, \infty))$, $x, w \in \mathbb{C}$, with $Z^\pm$, $\mathsf{X}^\pm$ as defined in* (105), (106)*, respectively, such that the complex variables $\alpha, z_1, z_2, x_1, x_2, x, w$ are in some yet to be determined neighborhood of the real line. Then*

$$Q_0^{(\alpha,3)}(Z^\pm) = 2\Gamma(\alpha+1)\Gamma(\alpha+1)\sum_{k=0}^{\infty}\frac{(\alpha+1)_k(\alpha+4)_k}{(\alpha+k)(4)_k}$$
$$\times\sum_{l=0}^{k}(\mp 1)^{k-l}(3+k-l)(\alpha+k+l)(\alpha+1)_l(z_1z_2)^{k-l}((z_1^2-1)(z_2^2-1))^{\frac{k+l}{2}}$$
$$\times\boldsymbol{Q}_{-k}^{(\alpha+k+l,3+k-l)}(2z_>^2-1)\boldsymbol{P}_{-k}^{(\alpha+k+l,3+k-l)}(2z_<^2-1)w^{k-l}P_l^{(\alpha-4,3+k-l)}(2w^2-1)C_{k-l}^3(x), \tag{198}$$

$$\mathsf{Q}_0^{(\alpha,3)}(\mathsf{X}^\pm) = \frac{1}{3}\Gamma(\alpha+1)\sum_{k=0}^{\infty}(-1)^k\frac{(\alpha+1)_k(\alpha+4)_k}{(\alpha+k)(4)_k}$$
$$\times\sum_{l=0}^{k}(\mp 1)^{k-l}(3+k-l)(\alpha+k+l)(x_1x_2)^{k-l}((1-x_1^2)(1-x_2^2))^{\frac{k+l}{2}}$$
$$\times\mathcal{Q}_{-k}^{(\alpha+k+l,3+k-l)}(2x_<^2-1)\mathsf{P}_{-k}^{(\alpha+k+l,3+k-l)}(2x_>^2-1)w^{k-l}P_l^{(\alpha-4,3+k-l)}(2w^2-1)C_{k-l}^3(x). \tag{199}$$

Proof. Setting $\beta = 3$ in Corollary 51 completes the proof. ■

The above calculations look almost trivial in that they are simply substitutions of the values of $\beta \in \{0, 1, 3\}$ and $\gamma = 0$ in the addition theorems given by Theorem 51. However, it should be understood that ordinarily these computations would be extremely difficult, particularly if one was to use the standard normalizations of the Jacobi functions. With standard normalizations of Jacobi functions these particular values, and in fact for values of integer parameters (α, β) and degrees γ, the Jacobi functions are not even defined. It is only because of the strategic choice of the particular normalization that we have chosen that the evaluation of these particular values becomes quite easy. We will further take advantage of these expansions in later publications.

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