Article

The Laguerre-type Classical Orthogonal Polynomials

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Abstract: A linear functional $\mathbf{u}$ is classical if there exist polynomials, ϕ and ψ, with $\deg\phi \le 2$, $\deg\psi = 1$, such that $\mathscr{D}(\phi(x)\mathbf{u}) = \psi(x)\mathbf{u}$, where $\mathscr{D}$ is certain differential, or difference, operator. The polynomials orthogonal with respect to the linear functional $\mathbf{u}$ are called classical orthogonal polynomials. In the theory of orthogonal polynomials, a correct characterization of the classical families is of great interest. In this work, on one hand, we present the different Laguerre-type families, which are those for which $\deg\phi = 1$, obtaining for all of them new algebraic identities such as structure formulas, orthogonality properties as well as new Rodrigues formulas; and for the other, we present a new characterization theorem for such Laguerre-type families.

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1. Introduction

Orthogonal polynomials $(p_n(x))_n$ associated with a measure on the real line, i.e.,

$$\int_{\mathbb{R}} p_m(x)p_n(x)d\mu(x) = d_n^2\,\delta_{mn},$$

satisfy a three-term recurrence equation

$$xp_n(x) = \alpha_n p_{n+1}(x) + \beta_n p_n(x) + \gamma_n p_{n-1}(x), \tag{1}$$

where $p_{-1}(x) = 0$, $p_0(x) = 1$ and, according to the Favard theorem (cf. [6, p. 21]), if $\gamma_n \neq 0$ for all $n \in \mathbb{N}$ this recurrence completely characterizes such a polynomial sequence.

Due to the property of orthogonality of such polynomial sequence, it is well-known the following relation between the recurrence coefficients:

$$\gamma_n = \alpha_{n-1}\frac{d_n^2}{d_{n-1}^2}, \quad n = 1, 2, \ldots, \tag{2}$$

where d_n^2 is the squared norm of p_n.

Such polynomial sequence is said to be classical if it is orthogonal with respect to a linear functional $\mathbf{u} : \mathbb{P} \to \mathbb{C}$ which fulfills the Pearson-type equation

$$\mathscr{D}(\phi(x)\mathbf{u}) = \psi(x)\mathbf{u}, \tag{3}$$

where ϕ is a polynomial of degree at most 2, ψ is a polynomial of degree 1, and $\mathscr{D}$ is the differential operator in the continuous case, the forward (Δ) or backward (∇) difference operator in the discrete case, and the Hahn operator $(\mathscr{D}_q)$ in the q-discrete case.

Remark 1. *Note that if a linear functional* $\mathbf{u}$ *fulfills* (3) *then it also fulfills the Pearson-type equation* [9]

$$\mathscr{D}^*(\phi^*(x)\mathbf{u}) = \psi^*(x)\mathbf{u},$$

where

- *In the continuous case* $\mathscr{D}^* = \mathscr{D}$, *so* $\phi^*(x) = \phi(x)$ *and* $\psi^*(x) = \psi(x)$.
- *In the discrete case* $\phi^*(x) = \phi(x) + \psi(x)$, $\psi^*(x) = \psi(x)$, $\Delta^* = \nabla$, $\nabla^* = \Delta$.
- *In the* q*-discrete case,* $\phi^*(x) = \phi(x) + (q-1)x\,\psi(x)$, $\psi^*(x) = q\psi(x)$, *and* $\mathscr{D}_q^* = \mathscr{D}_{q^{-1}}$.

Definition 1. *The polynomial sequence* $(p_n(x))_n$ *is a Laguerre-type Orthogonal Polynomial Sequence (in short Laguerre-type sequence) if it is classical, and* $\deg\phi = 1$ *or* $\deg\phi^* = 1$.

A fundamental role is played by the so-called characterization Theorems, i.e., the Theorems which collect those properties that completely define and characterize the classical orthogonal polynomials. One of the many ways to characterize a family of continuous classical polynomials (Hermite, Laguerre, Jacobi, and Bessel), which was first posed by R. Askey and proved by W. A. Al-Salam and T. S. Chihara [1] (see also [17]), is the structure relation

$$\phi\mathscr{D}p_n(x) = \widetilde{a}_n p_{n+1}(x) + \widetilde{b}_n p_n(x) + \widetilde{c}_n p_{n-1}(x), \tag{4}$$

where $\widetilde{c}_n \neq 0$.

A. G. Garcia et al. proved in [11] that the relation (4) also characterizes the discrete classical orthogonal polynomials (Hahn, Krawtchouk, Meixner, and Charlier polynomials) when the derivative is replaced by the forward difference operator Δ.

Later on, J. C. Medem et al. [18] characterized the orthogonal polynomials which belong to the q-Hahn class by a structure relation obtained from (4) replacing the derivative by the q-difference operator $\mathscr{D}_q$ (see also [2–4]). One of the most general characterization theorem for the q-polynomials in the q-quadratic lattice was done in [8].

The structure of this work is the following: in Section 2 we introduce some notations and definitions used throughout the paper. In Section 3 we present the main results about the Laguerre-type polynomials as well as the algebraic properties supporting the results presented.

2. Preliminaries

In this section we will give a brief survey of the operational calculus that we will use in the rest of the paper as well as some basic notation we need to prove the results.

2.1. Basic concepts and results

We adopt the following set notations: $\mathbb{N}_0 := \{0\} \cup \mathbb{N} = \{0, 1, 2, \ldots\}$, and we use the sets $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$ which represent the integers, real numbers, and complex numbers respectively. Let $\mathbb{P}$ be the linear space of polynomial functions in $\mathbb{C}$ (in the following we will refer to them as polynomials) with complex coefficients and $\mathbb{P}^*$ be its algebraic dual space, i.e., $\mathbb{P}^*$ is the linear space of all linear applications $\mathbf{u} : \mathbb{P} \to \mathbb{C}$. In the following, we will call the elements of $\mathbb{P}^*$ as functionals. In general, we will represent the action of a linear functional over a polynomial by

$$\langle \mathbf{u}, \pi \rangle, \quad \mathbf{u} \in \mathbb{P}^*,\ \pi \in \mathbb{P}.$$

Therefore, a functional is completely determined by a sequence of complex numbers $\langle \mathbf{u}, x^n \rangle = u_n$, $n = 0, 1, \ldots$, the so-called moments of the functional.

Definition 2. *Let* $\mathbf{u} \in \mathbb{P}^*$ *be a functional. We say that* $\mathbf{u}$ *is a* quasi-definite functional *if there exists a polynomial sequence* (p_n) *which is orthogonal with respect to* $\mathbf{u}$*, i.e.*

$$\langle \mathbf{u}, p_m p_n \rangle = k_n \delta_{mn}, \quad k_n \neq 0, \quad n = 0, 1, ...,$$

where δ_{mn} *is the Kronecker delta.*

In order to obtain our derived identities, we rely on properties of the shifted factorial, or Pochhammer symbol, $(a)_n$, and the q-shifted factorial, or q-Pochhammer symbol, $(a;q)_n$. For any $n \in \mathbb{N}_0$, $a, q \in \mathbb{C}$, the shifted factorial is defined as

$$(a)_n = a(a+1)\cdots(a+n-1),$$

the q-shifted factorial is defined as

$$(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}).$$

We will also use the common notational product conventions

$$(a_1, ..., a_k)_n = (a_1)_n \cdots (a_k)_n,$$
$$(a_1, ..., a_k; q)_n = (a_1;q)_n \cdots (a_k;q)_n.$$

The hypergeometric series ${}_rF_s$ is defined for $z \in \mathbb{C}$ such that $|z| < 1$, $s, r \in \mathbb{N}_0$, as [14, (1.4.1)]

$$ {}_rF_s\left(\begin{matrix} a_1, ..., a_r \\ b_1, ..., b_s \end{matrix}; z\right) := \sum_{k=0}^{\infty} \frac{(a_1, ..., a_r)_k}{(b_1, ..., b_s)_k} \frac{z^k}{k!}, \tag{5}$$

and he basic hypergeometric series ${}_r\phi_s$ is defined for $q, z \in \mathbb{C}$ such that $|q|, |z| < 1$, $s, r \in \mathbb{N}_0$, as [14, (1.10.1)]

$$ {}_r\phi_s\left(\begin{matrix} a_1, ..., a_r \\ b_1, ..., b_s \end{matrix}; q, z\right) := \sum_{k=0}^{\infty} \frac{(a_1, ..., a_r; q)_k}{(q, b_1, ..., b_s; q)_k} \left((-1)^k q^{\binom{k}{2}}\right)^{1+s-r} z^k. \tag{6}$$

Observe that both, hypergeometric and basic hypergeometric series, are entire functions of z for $s+1 > r$, are convergent for $|z| < 1$ for $s+1 = r$, and divergent unless it is terminating for $s+1 < r$.

Note that when we refer to a hypergeometric or basic hypergeometric function with *arbitrary argument* z, we mean that the argument does not necessarily depend on the other parameters, namely the a_j's, b_j's. However, for the arbitrary argument z, it very-well may be that the domain of the argument is restricted, such as for $|z| < 1$.

The next theorem [6, p. 8] is useful if one works with linear functional.

Theorem 1. *Let* $\mathbf{u} \in \mathbb{P}^*$ *be a functional with moments* (u_n)*. Then,* $\mathbf{u}$ *is quasi-definite if and only if the Hankel determinants* $H_n := \det(u_{i+j})_{i,j=0}^{n} \neq 0$, $n = 0, 1, ...$

2.2. Definition of the operators in $\mathbb{P}$ and $\mathbb{P}^*$

Next, we will define the backward and forward difference operators as well the so called q-derivative operator, or Hahn operator.

$$\begin{aligned} \Delta f(x) &= f(x+1) - f(x), \\ \nabla f(x) &= f(x) - f(x-1), \\ \mathscr{D}_q f(x) &= \frac{f(qx) - f(x)}{x(q-1)}, \quad q \neq 1,\ x \neq 0. \end{aligned}$$

Since the polynomial sequences depend on n, x and its parameters and we are going to focus more on the variable n along this work (n as a discrete variable) than in the variable x, we need to consider along the paper as well the operators Δ_n and ∇_n that are defined analogously as the operators Δ and ∇, i.e.

$$\nabla_n f(n;x) = f(n,x) - f(n-1,x), \qquad \Delta_n f(n;x) = f(n+1,x) - f(n,x).$$

Definition 3. *Let $\mathbf{u} \in \mathbb{P}^*$ and $\pi \in \mathbb{P}$, let $\Delta\mathbf{u}$, $\nabla\mathbf{u}$ and $\mathscr{D}_q^*\mathbf{u}$ be the linear functionals defined by*

$$\left\langle \frac{d}{dx}\mathbf{u}, \pi \right\rangle = -\langle \mathbf{u}, \pi' \rangle, \tag{7}$$

$$\langle \Delta\mathbf{u}, \pi \rangle = -\langle \mathbf{u}, \nabla\pi \rangle, \tag{8}$$

$$\langle \nabla\mathbf{u}, \pi \rangle = -\langle \mathbf{u}, \Delta\pi \rangle, \tag{9}$$

$$\langle \mathscr{D}_q^*\mathbf{u}, \pi \rangle = -q\langle \mathbf{u}, \mathscr{D}_q\pi \rangle. \tag{10}$$

Notice that we use the same notation for the operators on $\mathbb{P}$ and $\mathbb{P}^*$. Whenever it is not specified the linear space where an operator acts, it will be understood that it acts on the polynomial space $\mathbb{P}$.

3. The Laguerre-type polynomials

In this section, we are going to present all the identities of the different Laguerre-type families. First, we are going to show some theoretical aspects and results related to them.

Lemma 1. *Let $\mathbf{u} \in \mathbb{P}^*$ be a quasi-definite classical functional, let (p_n) be the polynomial sequence orthogonal with respect to $\mathbf{u}$. If (p_n) is a Laguerre-type Classical Orthogonal Polynomial Sequence then, there exists a numerical sequence $(\lambda_n)_n$ so that $(\lambda_n\, p_n)_n$ fulfills the recurrence relation (1) for which $\gamma_0 = 0$ and $\alpha_n + \beta_n + \gamma_n = \mathtt{c}$ for all $n \in \mathbb{N}$, where $\mathtt{c}$ is a root of $\phi(x)$, or $\phi^*(x)$.*

Proof. To prove this result it is enough to consider all the families of the Hypergeometric orthogonal polynomials scheme and the basic Hypergeometric orthogonal polynomials scheme (see e.g. [14,18,19]) that are of Laguerre-type, i.e., $\deg\phi = 1$ or $\deg\phi^* = 1$. These families are the Laguerre (`L`), Charlier (`C`), Meixner (`M`), big q-Laguerre (`bqL`), q-Meixner (`qM`), little q-Laguerre (`lqL`), q-Laguerre (`qL`), q-Charlier (`qC`), and the Stieltjes-Wigert (`SW`) polynomials.

Once we obtain all the Laguerre-type families it is enough to verify that the conditions established for each of the families (see Table 1) are satisfied. Observe that value of $p_n(\mathtt{c})$, where $\mathtt{c}$ is a zero of ϕ or ϕ^*, is known (see [15,20]), and these values are non-zero so one can define λ_n as $1/p_n(\mathtt{c})$. □

Remark 2. *Note that Lemma 1 is <u>not</u> a Characterization Theorem since other families of the basic Hypergeometric orthogonal polynomials scheme fulfill the condition about the recurrence coefficients,* *in short (RC), i.e. $\alpha_n + \beta_n + \gamma_n$ is a constant, however they are not Laguerre-type families, for example, the Askey-Wilson polynomials (see [14, (14.1.4)]).*

In the next result we write the recurrence relation as a structure-type relation on n.

Lemma 2. *Let $(p_n)_n$ be a Laguerre-type polynomial sequence. For any $x \in \mathbb{C}$ the recurrence relation (1) can be written as:*

$$\phi(x)p_n(x) = \alpha_n \Delta_n p_n(x) - \gamma_n \Delta_n p_{n-1}(x), \tag{11}$$

if $\deg \phi = 1$*; and*

$$\phi^*(x)p_n(x) = \alpha_n \Delta_n p_n(x) - \gamma_n \Delta_n p_{n-1}(x),$$

if $\deg \phi^* = 1$.

Proof. Let us assume that $\deg \phi = 1$, i.e., $\phi(x) = x - \mathsf{c}$, then by Lemma 1 we have that the coefficients of the recurrence relation (1) fulfill $\alpha_n + \beta_n + \gamma_n = \mathsf{c}$, then rewriting the recurrence assuming this relation the result holds. The $\deg \phi^* = 1$ case is analogous. □

Remark 3. *For the sake of convenience, we are going to replace $x - \mathsf{c}$ by $\phi(x)$ assuming that $\deg \phi = 1$. In the case that $\deg \phi \neq 1$ and $\deg \phi^* = 1$, then one must replace ϕ by ϕ^* in the further results since such identities and results hold similarly.*

We write the recurrence relation in the next result as a Sturm-Liouville form difference equation on n.

Lemma 3. *Let $(p_n)_n$ be a Laguerre-type polynomial sequence. For any $x \in \mathbb{C}$ the recurrence relation (1) can be written as:*

$$\phi(x)p_n(x) = d_n^2 \nabla_n \frac{\gamma_n}{d_n^2} \Delta_n p_n(x), \tag{12}$$

$$= d_n^2 \Delta_n \frac{\alpha_n}{d_n^2} \nabla_n p_n(x). \tag{13}$$

Proof. Starting from (11), using (2) and taking into account the definition of ∇_n and Δ_n the results follow. □

Theorem 2. *Let $(p_n)_n$ be a Laguerre-type polynomial sequence. For any $x \in \mathbb{C}$, the sequence $(\nabla_n p_n(x))$ is orthogonal with respect to α_n/d_n^2, and the sequence $(\Delta_n p_n(x))$ is orthogonal with respect to γ_n/d_n^2.*

Proof. Let us fix $x \in \mathbb{C}$ such that $\omega(x) \neq 0$. By Korovkin's Theorem (see [16] or [12, Theorem 2.1]) we have the following closure relation:

$$\sum_{n=0}^{\infty} \frac{p_n(x)p_n(y)}{d_n^2} = \frac{1}{\omega(x)} \delta(x-y),$$

where ω is the weight function and both the left- and right-hand sides should be treated as distributions. From this expression, using (13) and Abel's lemma on partial sums [13, p. 313] for $y \neq x$ we obtain

$$0 = \sum_{n=0}^{\infty} \frac{\phi(x)p_n(x)p_n(y)}{d_n^2} = \sum_{n=0}^{\infty} \left(\Delta_n \frac{\alpha_n}{d_n^2} \nabla_n p_n(x) \right) p_n(y) = -\sum_{n=0}^{\infty} \frac{\alpha_n}{d_n^2} \nabla_n p_n(x) \nabla_n p_n(y).$$

Moreover,

$$\sum_{n=0}^{\infty} \frac{\alpha_n}{d_n^2} \nabla_n p_n(x) \nabla_n p_n(x) = \frac{\phi(x)}{\omega(x)}.$$

The proof for the sequence $(\Delta_n p_n(x))$ is analogous. Hence the result follows. □

Remark 4. *Since we have the data for the Laguerre-type polynomials it is a straightforward calculation to check that* $1/d_{-1}^2 = 0$ *(since* $\gamma_0 = 0$*) and*

$$\lim_{n\to\infty} \frac{\alpha_n}{d_n^2} = 0.$$

Theorem 3 (Characterization Theorem)**.** *Let* $(p_n)_n$ *be an orthogonal polynomial sequence with respect to* ω*, such that* $\mathscr{D}(\phi\omega) = \psi\omega$*, with* $\deg\phi = 1$*. For any* $x \in \mathbb{C}$ *such that* $\omega(x) \neq 0$*, the following statements are equivalent:*

1. $(p_n)_n$ *is a Laguerre-type polynomial sequence.*
2. *The polynomials sequence* $(\nabla_n p_n(x))$ *is orthogonal with respect to* α_n/d_n^2*.*
3. *The polynomial sequence* $(p_n(x))$ *fulfills the second order difference equation*

$$\phi(x)p_n(x) = \alpha_n \Delta_n p_n(x) - \gamma_n \Delta_n p_{n-1}(x),$$

which is equal to its structure-type relation.

4. *The polynomial sequence* $(p_n(x))$ *satisfies the Sturm-Liouville difference equations*

$$\phi(x)p_n(x) = d_n^2 \nabla_n \frac{\gamma_n}{d_n^2} \Delta_n p_n(x) = d_n^2 \Delta_n \frac{\alpha_n}{d_n^2} \nabla_n p_n(x).$$

4. The families

Along this section we present several identities related to the different Laguerre-type families. Since there are some relation among them (see Figure 1) we will present such identities only for some of the families in order to avoid duplicities. We consider the Laguerre, Meixner, Charlier, big q-Laguerre, little q-Laguerre and Stieltjes-Weigert cases.

Before presenting the main results let us show the relations between the families we are going to consider with respect to the rest of the Laguerre-type families (see [7, p. 20], [14, remark p. 526]):

$$p_n(x; a, b, 1/q) = \frac{1}{(q/b; q)_n} M_n(xq/a; 1/a, -b; q), \tag{14}$$

$$p_n(x; q^\alpha | 1/q) = \frac{(q; q)_n}{(q^{\alpha+1}; q)_n} L_n^{(\alpha)}(-x; q), \tag{15}$$

$$L_n^{(\alpha)}(x; q) = \frac{1}{(q; q)_n} C_n(-x; -q^{-\alpha}; q), \tag{16}$$

$$S_n(x/a, 1/q) = {}_2\phi_0\left(q^{-n}, 0; -; q, -\frac{x}{a}\right) =: l_n(x; a), \tag{17}$$

where the 0-Laguerre/Bessel polynomials $(l_n(x; a))$ (see [5], [19, p. 244]).

4.1. The Laguerre poynomials

The Laguerre polynomials can be written in terms of hypergeometric series as [14, §9.12]

$$L_n^{(\alpha)}(x) = \frac{(\alpha+1)_n}{n!} \, {}_1F_1\left(\begin{matrix} -n \\ \alpha+1 \end{matrix}; x\right).$$

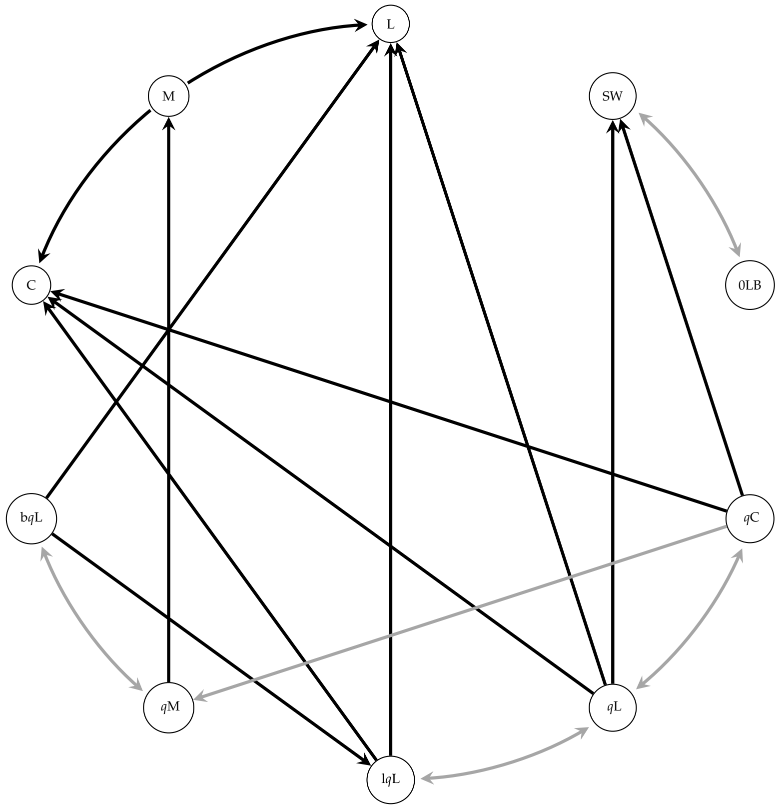

Figure 1. Relations between the Laguerre-type families. The gray lines are the particular cases. The black lines are the limiting cases.

In this case $\phi(x) = x$, i.e., this is the a Laguerre-type family where the zero of ϕ is $\mathsf{c} = 0$, and $L_n^{(\alpha)}(\mathsf{c}) = (\alpha+1)_n/n!$. Taking this into account we can state the following result.

Lemma 4. *For any $\alpha \in \mathbb{C}$ and any $n \in \mathbb{N}_0$, the following identities hold:*

$$
\begin{aligned}
\nabla_n L_n^{(\alpha)}(x) &= L_n^{(\alpha-1)}(x), && (18)\\
\alpha L_n^{(\alpha)}(x) - (n+1)\Delta_n L_n^{(\alpha)}(x) &= x L_n^{(\alpha+1)}(x), && (19)\\
\alpha L_n^{(\alpha)}(x) - (n+\alpha)\nabla_n L_n^{(\alpha)}(x) &= x L_{n-1}^{(\alpha+1)}(x), && (20)\\
\nabla_\alpha L_n^{(\alpha)}(x) &= L_{n-1}^{(\alpha)}(x), && (21)\\
(n+\alpha-x) L_n^{(\alpha)}(x) - (\alpha+n)\nabla_\alpha L_n^{(\alpha)}(x) &= (n+1) L_{n+1}^{(\alpha-1)}(x), && (22)\\
(n+1+\alpha-x) L_n^{(\alpha)}(x) - x\Delta_\alpha L_n^{(\alpha)}(x) &= (n+1) L_{n+1}^{(\alpha)}(x), && (23)
\end{aligned}
$$

where $\Delta_\alpha f(x,\alpha) = f(x,\alpha+1) - f(x,\alpha)$, and $\nabla_\alpha f(x,\alpha) = \Delta_\alpha f(x,\alpha-1)$.

Proof. All these identities can be checked by identifying the polynomial coefficients on the left and right-hand sides of each identity. Hence, the results follow. □

A direct consequence of the former result is stated as follows.

Theorem 4. *For any $\alpha \in \mathbb{C}$, any $x \in \mathbb{C}$, $x \neq 0$, and any $n \in \mathbb{N}_0$, the polynomial sequence $\left(L_n^{(\alpha+k)}(x)\right)_k$ is orthogonal with respect to certain moment functional.*

Proof. By combining (21), (22) and (23) we have the following second order difference equations:

$$nL_n^{(\alpha)}(x) = -(\alpha+n)\nabla_\alpha \Delta_\alpha L_n^{(\alpha)}(x) + (\alpha+n-x)\Delta_\alpha L_n^{(\alpha)}(x) \quad (24)$$

$$= -x\Delta_\alpha \nabla_\alpha L_n^{(\alpha)}(x) + (\alpha+n-x)\nabla_\alpha L_n^{(\alpha)}(x), \quad (25)$$

which is connected with the Charlier polynomials case (see [14, (9.14.5)]). By using the theory of Sturm-Liouville, the result holds. □

Remark 5. *Observe that one can construct, under certain condition on α and x, certain integral representation for such moment functional.*

From Lemma 4 we can deduce some new identities related to the Laguerre polynomials.

Theorem 5. *For any $\alpha \in \mathbb{C}$, and any $n, k \in \mathbb{N}_0$, the following Rodrigues-type identities hold:*

$$L_n^{(\alpha+k)}(x) = \frac{(\alpha+k)_n}{x^k}(\alpha+k)(\alpha+k-1)\nabla_\alpha \cdots (\alpha+1)\alpha\nabla_\alpha \frac{L_{n+k}^{(\alpha)}(x)}{(\alpha)_{n+k}}, \quad (26)$$

$$L_n^{(\alpha+k)}(x) = \frac{(-1)^k(\alpha+1)_{n+k}}{n!x^k}\nabla_n^k \frac{(n+k)!}{(\alpha+1)_{n+k}} L_{n+k}^{(\alpha)}(x), \quad (27)$$

$$L_n^{(\alpha)}(x) = (-1)^k \frac{(n+1)_\alpha}{x^\alpha}\Delta_\alpha^k \frac{x^\alpha}{(n-k+1)_k} L_{n-k}^{(\alpha)}(x), \quad (28)$$

$$L_n^{(\alpha)}(x) = \Delta_n^k L_{n-k}^{(\alpha+k)}(x), \quad (29)$$

$$L_n^{(\alpha+k)}(x) = \Delta_\alpha^k L_{n+k}^{(\alpha)}(x). \quad (30)$$

Proof. The first identity holds by mathematical induction on k after a straightforward simplification and using (20) written in the following way:

$$\frac{n!}{(\alpha+k+1)_n} L_n^{(\alpha+k)}(x) = \frac{(\alpha+k)(\alpha+k-1)}{x(n+1)}\nabla_\alpha \frac{(n+1)!}{(\alpha+k)_{n+1}} L_{n+1}^{(\alpha+k-1)}(x).$$

The second identity holds by mathematical induction on k after a straightforward simplification and using (22) written in the following way:

$$\frac{n!}{(\alpha+k+1)_n} L_n^{(\alpha+k)}(x) = -\frac{\alpha+k}{x}\nabla_n \frac{(n+1)!}{(\alpha+k)_{n+1}} L_{n+1}^{(\alpha+k-1)}(x).$$

The third identity holds by mathematical induction on k after a straightforward simplification and using (23) written in the following way:

$$x^\alpha L_n^{(\alpha)}(x) = -(n+1)_\alpha \Delta_n \frac{x^\alpha}{(n)_\alpha} L_{n-1}^{(\alpha)}(x).$$

The fourth and the fifth relation hold from (18) and (21). □

The last result, but not least interesting, concerning the operators associated with the Laguerre polynomials is as follows.

Proposition 1. *The laguerre polynomials fulfill the following identity:*

$$(2+a-x)L_n^{(\alpha)}(x) + (-4-2a+3x)(L_n^{(\alpha)}(x))' + (2+a-3x)(L_n^{(\alpha)}(x))'' + x(L_n^{(\alpha)}(x))''' = (n+1)L_{n+1}^{(\alpha+}$$

Proof. The identity follows using the identities:

$$L_n^{(\alpha)}(x) - (L_n^{(\alpha)}(x))' = L_n^{(\alpha+1)}(x), \tag{31}$$

$$x(L_n^{(\alpha)}(x))' - (x-\alpha)L_n^{(\alpha)}(x) = (n+1)L_{n+1}^{(\alpha-1)}(x), \tag{32}$$

applying the first twice, and later the second one once mapping $\alpha \mapsto \alpha + 2$. □

4.2. The Charlier polynomials

The Charlier polynomials can be written in terms of hypergeometric series as [14, §9.14]

$$C_n(x;a) = {}_2F_0\left(\begin{matrix} -n, -x \\ - \end{matrix}; -\frac{1}{a}\right).$$

In this case $\phi(x) = x$ and $\phi^*(x) = a$, i.e., $\mathrm{c} = 0$, and $C_n(\mathrm{c};a) = 1$. Taking this into account we can state the following result.

Lemma 5. *For any $a \in \mathbb{C}$ and any $n \in \mathbb{N}_0$, the following identities hold:*

$$a\Delta_n C_n(x;a) = -\,x\,C_n(x-1;a), \tag{33}$$

$$nC_n(x;a) + a(C_n(x;a))' = (n+1)C_{n-1}(x;a), \tag{34}$$

$$(a-n)C_n(x;a) + n\nabla_n C_n(x;a) = a\,C_n(x+1;a), \tag{35}$$

$$(a-n-1)C_n(x;a) + a\Delta_n C_n(x;a) = a\,C_{n+1}(x+1;a). \tag{36}$$

Proof. The ientity (34) is a direct consequences of the identity presented in Remark in [14, p. 249]

$$\frac{(-a)^n}{n!} C_n(x;a) = L_n^{(x-n)}(a).$$

The other identities can be checked by identifying the polynomial coefficients on the left and right-hand sides of each identity. Hence, the results follow □

From Lemma 5 we can deduce some new identities related to the Charlier polynomials.

Theorem 6. *For any $a \in \mathbb{C}$, and any $n, k \in \mathbb{N}_0$, the following Rodrigues-type identities hold:*

$$C_n(x;a) = \frac{(-a)^k}{(x+1)_k} \Delta_n^k C_n(x+k;a), \tag{37}$$

$$C_n(x;a) = \frac{n!}{a^n} \Delta_n^k \frac{a^{n-k}}{(n-k)!} C_{n-k}(x-k;a). \tag{38}$$

Proof. The first identity is a direct consequence of (33); and the second is due the identity:

$$(n+1)!\Delta_n \frac{a^n}{n!} C_n(x;a) = a^{n+1} C_{n+1}(x+1;a).$$

□

4.3. The Meixner polynomials

The Meixner polynomials can be written in terms of hypergeometric series as [14, §9.10]

$$M_n(x;\beta,c) = {}_2F_1\left(\begin{matrix} -n, -x \\ \beta \end{matrix}; 1 - \frac{1}{c}\right).$$

In this case $\phi(x) = x$ and $\phi^*(x) = c(x+\beta)$, therefore we must consider two cases, $\mathsf{c}_1 = 0$, and $\mathsf{c}_2 = -\beta$, for which we have $M_n(\mathsf{c}_1;\beta,c) = 1$, and $M_n(\mathsf{c}_2;\beta,c) = 1/c^n$. Taking into account Lemma 1 we can state the next result.

Lemma 6. *The polynomial sequence* $(M_n(x;\beta,c)/M_n(\mathsf{c}_2;\beta,c))_n$ *fulfills the recurrence relation*

$$(x+\beta)p_n(x) = \alpha_n^{\mathtt{M}} p_{n+1}(x) - (\alpha_n^{\mathtt{M}} + \gamma_n^{\mathtt{M}})p_n(x) + \gamma_n^{\mathtt{M}} p_{n-1}(x),$$

with initial condition $p_0(x) = 1$,

$$\alpha_n^{\mathtt{M}} = \frac{n+\beta}{c-1}, \qquad \gamma_n^{\mathtt{M}} = \frac{nc}{c-1},$$

and fulfills the second order difference equation

$$(x+\beta)p_n(x) = d_n^2 \Delta_n \frac{\gamma_n^{\mathtt{M}}}{d_n^2} \nabla_n p_n(x) = d_n^2 \nabla_n \frac{\alpha_n^{\mathtt{M}}}{d_n^2} \Delta_n p_n(x) = \gamma_n^{\mathtt{M}} \nabla_n \Delta_n p_n(x) + (\alpha_n^{\mathtt{M}} - \gamma_n^{\mathtt{M}}) \Delta_n p_n(x).$$

As in the Laguerre polynomial case, the next result follows.

Lemma 7. *For any* $\beta, c \in \mathbb{C}$, $\beta \notin \{0,2\}$, $c \notin \{0,1\}$, *and any* $n \in \mathbb{N}_0$, *the following identities hold:*

$$\frac{\beta c}{c-1}\, \Delta_n M_n(x;\beta,c) = x\, M_n(x-1;\beta+1,c), \tag{39}$$

$$M_n(x;\beta,c) + \frac{c}{c-1} \Delta_n M_n(x;\beta,c) = \frac{x+\beta}{\beta}\, M_n(x;\beta+1,c), \tag{40}$$

$$M_n(x;\beta,c) + \frac{1}{c-1} \nabla_n M_n(x;\beta,c) = \frac{x+\beta}{\beta}\, M_{n-1}(x;\beta+1,c), \tag{41}$$

$$\frac{c\beta(1-\beta)}{c-1}\, \nabla_\beta M_n(x;\beta,c) = xn M_{n-1}(x-1;\beta+1,c), \tag{42}$$

$$\begin{aligned}\frac{\beta(\beta-1)c}{(\beta+n)(c-1)} M_{n+1}(x+1;\beta-1,c) =& \left(x+\beta+\frac{\beta(\beta-1)}{(\beta+n)(c-1)}\right) M_n(x;\beta,c) \\ &+ (x+\beta)\Delta_\beta M_n(x;\beta,c),\end{aligned} \tag{43}$$

$$\begin{aligned}\frac{(\beta-1)(\beta-2)c}{(\beta-1+n)(c-1)} M_{n+1}(x+1;\beta-2,c) =& \left(x+\beta-1+\frac{(\beta-1)(\beta-2)}{(\beta-1+n)(c-1)}\right) M_n(x;\beta,c) \\ &- \frac{(\beta-1)(\beta-2)}{(\beta-1+n)(c-1)} \nabla_\beta M_n(x;\beta,c),\end{aligned} \tag{44}$$

$$\frac{\partial}{\partial c} c^n (\beta)_n M_n(x;\beta,c) = \frac{n(x+\beta)}{c+n+\beta} c^n (\beta+1)_n M_{n-1}(x;\beta+1,c), \tag{45}$$

$$\begin{aligned}c(1-\beta)c^n(\beta)_n \beta M_{n+1}(x;\beta-1,c) =& c(1-c)\frac{\partial}{\partial c} c^n (\beta)_n M_n(x;\beta,c) \\ &- ((c-1)x + n - (n+1)c + c\beta) c^n (\beta)_n \beta M_n(x;\beta,c)\end{aligned} \tag{46}$$

Proof. All these identities can be checked by identifying the polynomial coefficients on the left and right-hand sides of each identity and with the help of Wolfram Mathematica 13. □

A direct consequence is the fact this polynomial sequence is orthogonal with respect to the parameter β.

Theorem 7. *For any* $\beta, c \in \mathbb{C}$, *any* $x \in \mathbb{C}$, $x \notin \{0, -\beta\}$, *and any* $n \in \mathbb{N}_0$, *the polynomial sequence* $\big(M_n(x;\beta+k,c)\big)_k$ *is orthogonal with respect to certain moment functional.*

Proof. By combining (42), (43) and (44) we have the following second order difference equations:

$$nxM_n(x;\beta,c)=\frac{\beta(\beta-1)}{c-1}\nabla_\beta\Delta_\beta M_n(x;\beta,c)-\left(\frac{\beta(\beta-1)}{c-1}+(\beta+x)(\beta+n)\right)\Delta_\beta M_n(x;\beta,c), \quad (47)$$

which is connected with the Continuous Hahn polynomials case (see [14, (9.4.5)]). By using the theory of Sturm-Liouville, the result holds. □

Remark 6.
- *Note that in [10] the authors extended the orthogonality relations for the Meixner polynomials thorough the orthogonality relations of the Continuous Hahn Polynomials.*
- *Observe that one can construct, under certain condition on the parameters, n and x, certain integral representation for such moment functional.*

From Lemma 7 we can deduce some new identities related to the Meixner polynomials.

Theorem 8. *For any $\beta, c\in\mathbb{C}$, and any $n,k\in\mathbb{N}_0$, the following Rodrigues-type identities hold:*

$$M_n(x;\beta+k,c)=\frac{(\beta)_k c^k}{(c-1)^k(x+1)_k}\nabla_n^k M_{n+k}(x;\beta,c), \quad (48)$$

$$M_n(x;\beta+k,c)=\frac{(\beta)_k}{(1-c)^k(x+\beta)_k(-c)^n}\Delta_n^k(-c)^{n+k}M_{n+k}(x;\beta,c), \quad (49)$$

$$M_n(x;\beta+k,c)=\frac{c^k(\beta+k-1)(2-k-\beta)}{(c-1)^k(x+1)_k(n+1)_k}\nabla_\beta\cdots\beta(1-\beta)\nabla_\beta M_{n+k}(x+k;\beta,c), \quad (50)$$

$$c^n(\beta+k)_nM_n(x;\beta+k,c)=\frac{(c+n+\beta+k)}{(n+1)_k(x+\beta)_k}\frac{\partial}{\partial c}\cdots(c+n+\beta+k)\frac{\partial}{\partial c}c^{n+k}(\beta)_{n+k}M_{n+k}(x;\beta,c). \quad (51)$$

Proof. The first identity holds by mathematical induction on k after a straightforward simplification and using (39). The second identity holds by mathematical induction on k after a straightforward simplification and using (41) written in the following way:

$$\Delta_n(-c)^nM_n(x;\beta,c)=\frac{x+\beta}{\beta}(1-c)(-c)^nM_{n-1}(x;\beta+1,c).$$

The third identity holds by mathematical induction on k after a straightforward simplification and using (42). The fourth and the fifth relation hold from (45). □

The last result concerning the operators associated with the Meixner polynomials is as follows.

Proposition 2. *The following identity holds:*

$$\begin{aligned}&\frac{(\beta c+cx+c-x)}{c}M_n(\beta,c,x)+\frac{(2\beta c^2+2c^2x+2c^2-cx-x)}{(c-1)c}\nabla M_n(\beta,c,x)\\&+\frac{c(-2\beta+3\beta c+3cx+3c-3x-2)}{(c-1)^2}\nabla^2M_n(\beta,c,x))+\frac{c^2(\beta+x+1)}{(c-1)^2}\nabla^3M_n(\beta,c,x)\\&=\frac{(\beta+n)(\beta+n+1)}{\beta}M_{n+1}(\beta+1,c,x).\end{aligned}$$

Proof. The proof follows after using an algorithm written in Wolfram Mathematica 13. □

4.4. The big q-Laguerre polynomials

The big q-Laguerre polynomials can be written in terms of basic hypergeometric series as [14, §14.11]

$$p_n(x;a,b;q) = {}_3\phi_2\left(\begin{matrix} q^{-n}, 0, x \\ aq, bq \end{matrix}; q, q\right).$$

In this case $\phi(x) = (x-aq)(x-bq)$ and $\phi^*(x) = abq(1-x)$, i.e., $\mathrm{c}=1$, and $p_n(\mathrm{c};a,b;q) = 1$. Observe these polynomials are symmetric in the parameters a and b. Taking this into account we can state the following result.

Lemma 8. *For any $a, b \in \mathbb{C}$, and any $n \in \mathbb{N}_0$, the following identities hold:*

$$\phi(1)q^n\,\Delta_n p_n(x;a,b;q) = (x-1)\,p_n(xq;aq,bq;q), \tag{52}$$

$$\begin{aligned}(-1+a)(-1+b)q^{n+1}\,p_{n+1}(x/q;a/q,b/q;q) =& (abq^{n+1} - aq^{n+1} - bq^{n+1} + 1)p_n(x;a,b;q) \\ &+ \alpha_n^{\mathtt{bqL}}\Delta_n p_n(x;a,b;q),\end{aligned} \tag{53}$$

$$\begin{aligned}(-1+a)(-1+b)q^{n+1}\,p_n(x/q;a/q,b/q;q) =& q(abq^n - aq^n - bq^n + 1)p_n(x;a,b;q) \\ &+ q^n\gamma_n^{\mathtt{bqL}}\nabla_n p_n(x;a,b;q),\end{aligned} \tag{54}$$

$$(a-1)\phi(1)\,\mathscr{D}_{q,a}\,p_n(x;a,b;q) = \frac{a(1-q^n)}{q^{n-1}}(x-1)p_{n-1}(xq;aq,bq;q), \tag{55}$$

$$\begin{aligned}(a-1)bq^{n+1}(aq-1)p_n(x;a/q,b;q) =& (aq^{n+1}-1)(x-aq)\mathscr{D}_{q,a}\,p_n(x;a,b;q) \\ &+((1-aq)(x-(a-b+ab)q^{n+1}) + (aq^{n+1}-1)(x-aq))p_n(x;a,b;q),\end{aligned} \tag{56}$$

$$\begin{aligned}(a-1)(aq-1)(bq^{n+1}-1)\,p_{n+1}(x;a/q,b;q) =& (1-a)b(aq-1)q^{n+1}\mathscr{D}_{1/q,a}p_n(x;a,b;q) \\ &+ (aq-1)(x+abq^{n+1}-a-bq^{n+1})p_n(x;a,b\end{aligned} \tag{57}$$

where $\alpha_n^{\mathtt{bqL}}$ and $\gamma_n^{\mathtt{bqL}}$ are the big q-Laguerre recurrence relation coefficients (see Table 1), and

$$\mathscr{D}_{q,a}f(x,a) = \frac{f(x,qa) - f(x,a)}{a(q-1)}.$$

Proof. All these identities can be checked by identifying the polynomial coefficients on the left and right-hand sides of each identity and with the help of Wolfram Mathematica 13. □

A direct consequence is the fact this polynomial sequence is orthogonal with respect to the parameters a and b.

Theorem 9. *For any $a, b \in \mathbb{C}$, any $x \in \mathbb{C}$, $x \notin \{1, aq, bq\}$, and any $n \in \mathbb{N}_0$, the polynomial sequences $\left(p_n(x;aq^k,b;q)\right)_k$ and $\left(p_n(x;a,bq^k;q)\right)_k$ are orthogonal with respect to certain moment functional.*

Proof. By combining (55), (56) and (57) we have the following second order difference equations:

$$(x-1)a(1-q^n)p_n(x;a,b;q) = \mathrm{t}_n(a)\mathscr{D}_{1/q,a}\mathscr{D}_{q,a}p_n(x;a,b;q) - (\mathrm{t}_n(a) - \mathrm{t}_n^*(a))\mathscr{D}_{q,a}p_n(x;a,b;q), \tag{58}$$

$$= \mathrm{t}_n^*(a)\mathscr{D}_{q,a}\mathscr{D}_{1/q,a}p_n(x;a,b;q) - (\mathrm{t}_n(a) - \mathrm{t}_n^*(a))\mathscr{D}_{1/q,a}p_n(x;a,b;q), \tag{59}$$

where $\mathrm{t}_n(a) = (aq^{n+1} - 1)(x - aq)$, and $\mathrm{t}_n^*(a) = (a-1)bq^{n+1}(aq-1)$, which is connected with the big q-Jacobi polynomials case (see [14, (14.5.5)]). By using the theory of Sturm-Liouville, the result holds. The result also holds for the parameter b by symmetry. □

From Lemma 8 we can deduce some new identities related to the big q-Laguerre polynomials.

Theorem 10. *For any $a, b \in \mathbb{C}$, and any $n, k \in \mathbb{N}_0$, the following Rodrigues-type identities hold:*

$$p_n(q^k x; aq^k, bq^k; q) = \frac{(aq, bq; q)_k}{(x; q)_k} \nabla_n^k p_{n+k}(x; a, b; q), \tag{60}$$

$$p_n(q^k x; aq^k, bq^k; q) = \frac{q^{nk} q^{\binom{k+1}{2}} (bq; q)_k}{(q^{n+1}, x; q)_k} \frac{(aq^{k-1} - 1)(1 - aq^k)}{aq^k} \mathscr{D}_{q,a} \cdots \frac{(a-1)(1-aq)}{aq} \mathscr{D}_{q,a} p_{n+k}(x; a, b; q) \tag{61}$$

Proof. The identities holds by mathematical induction on k after a straightforward simplification and using (52) and (55). □

The last result concerning the operators associated with the big q-Laguerre polynomials is as follows.

Proposition 3. *The following identity holds:*

$$\frac{(cq^{n+1} - 1)(cq^{n+2} - 1)(\beta q^{n+1} - 1)(\beta q^{n+2} - 1)}{q^{4n}(cq-1)(\beta q - 1)} p_{n+1}(q^4 x, aq, bq; q) = \sum_{k=0}^{4} p_k(x) \mathscr{D}_q^k p_n(x; a, b; q)$$

where the polynomial coefficients are

$$\begin{aligned}
p_0(x) =& \frac{1}{q}\big((q^4x-1)(a+b+aq+bq+abq) + (a+b)(1+q)(aq^2-1)(bq^2-1)\big),\\
p_1(x) =& \frac{(q-1)}{q^2}\big((q^4x-1)(ab+a^2q+2abq+a^2bq+b^2q+ab^2q+abq^2+a^2bq^2+ab^2q^2)\\
&+ (aq^2-1)(bq^2-1)(ab+a^2q+2abq+b^2q+abq^2)\big),\\
p_2(x) =& \frac{ab(q-1)^2}{q^2}\big((q^4x-1)(a+b+ab+aq+a^2q+bq+2abq+b^2q+abq^2)\\
&+ (a+b)(1+q)(aq^2-1)(bq^2-1)\big),\\
p_3(x) =& \frac{a^2b^2(q-1)^3}{q^2}\big((q^4x-1)(1+a+b+aq+bq) + (aq^2-1)(bq^2-1)\big),\\
p_4(x) =& \frac{a^3b^3(q-1)^4}{q^2}(q^4x-1).
\end{aligned}$$

Proof. The proof follows after using an algorithm written in Wolfram Mathematica 13. □

4.5. *The little q-Laguerre/Wall polynomials*

The little q-Laguerre polynomials can be written in terms of basic hypergeometric series as [14, §14.20]

$$p_n(x; a|q) = {}_2\phi_1\left(\begin{matrix} q^{-n}, 0 \\ aq \end{matrix}; q, qx\right).$$

In this case $\phi(x) = (1-x)x$ and $\phi^*(x) = ax$, i.e., $\mathrm{c} = 0$, and $p_n(\mathrm{c}; a|q) = 1$. Taking this into account we can state the following result.

Lemma 9. *For any $a \in \mathbb{C}$, and any $n \in \mathbb{N}_0$, the following identities hold:*

$$
\begin{aligned}
\frac{q^{1-n}\,x}{aq-1}p_{n-1}(x;aq|q) &= \nabla_n p_n(x;a|q), && (62)\\
(1-a)(x-1)p_n(x/q;a/q|q) &= (aq^{n+1}-1)\Delta_n p_n(x;a|q)-(ax+1-a)p_n(x;a|q), && (63)\\
(1-a)(x-1)p_{n-1}(x/q;a/q|q) &= (ax+(q^n-1)a)\nabla_n p_n(x;a|q)-(ax+1-a)p_n(x;a|q), && (64)\\
\frac{a(1-q^n)x}{q^{n-1}(1-a)(1-aq)}p_{n-1}(x;aq|q) &= \mathscr{D}_{1/q,a}\, p_n(x;a|q), && (65)\\
(a-1)q^n(aq-1)p_n(x;a/q|q) &= (aq^{n+1}-1)x\mathscr{D}_{1/q,a}\, p_n(x;a|q)\\
&\quad + (aq(q^n-1)x+q^n(aq-1)(a-1))p_n(x;a|q), && (66)\\
a(a-1)q^{n-1}(q^n-1)p_{n-1}(x;a/q|q) &= (a-1)(x+aq^{2n-1}-aq^{n-1})\mathscr{D}_{q,a}\, p_n(x;a|q)\\
&\quad + (a(q^n-1)x+(1-a)aq^{n-1}(q^n-1))p_n(x;a|q). && (67)
\end{aligned}
$$

Proof. All these identities can be checked by identifying the polynomial coefficients on the left and right-hand sides of each identity and with the help of Wolfram Mathematica 13. □

A direct consequence is the fact this polynomial sequence is orthogonal with respect to the parameter a.

Theorem 11. *For any $a \in \mathbb{C}$, any $x \in \mathbb{C}$, $x \notin \{1, aq\}$, and any $n \in \mathbb{N}_0$, the polynomial sequence $\left(p_n(x;aq^k|q)\right)_k$ is orthogonal with respect to certain moment functional.*

Proof. By combining (55), (56) and (57) we have the following second order difference equations:

$$
\begin{aligned}
aqx(q^n-1)p_n(x;a|q) &= \mathsf{s}_n(a)\mathscr{D}_{1/q,a}\mathscr{D}_{q,a}p_n(x;a|q) - (\mathsf{s}_n(a)-\mathsf{s}_n^*(a))\mathscr{D}_{q,a}p_n(x;a|q) && (68)\\
&= \mathsf{s}_n^*(a)\mathscr{D}_{q,a}\mathscr{D}_{1/q,a}p_n(x;a|q) - (\mathsf{s}_n(a)-\mathsf{s}_n^*(a))\mathscr{D}_{1/q,a}p_n(x;a|q) && (69)
\end{aligned}
$$

where $\mathsf{s}_n(a) = (a-1)q^n(aq-1)$, and $\mathsf{s}_n^*(a) = (1-aq^{n+1})x$, which is connected with the big q-Laguerre polynomials case (see [14, (14.11.5)]). By using the theory of Sturm-Liouville, the result holds. □

The last result concerning the operators associated with the little q-Laguerre polynomials is as follows.

Proposition 4. *The following identity holds:*

$$
\frac{(aq^{n+1}-1)(aq^{n+2}-1)}{q^{3n}(q-1)(aq-1)}p_{n+1}(q^4x, aq|q) = \sum_{k=0}^{4} \mathsf{p}_k(x)\mathscr{D}_q^k p_n(x;a|q)
$$

where the polynomial coefficients are

$$
\begin{aligned}
\mathsf{p}_0(x) =& \frac{1}{q^3}\left(q^4x(aq^3-aq-1-q-q^2)+(aq^2-1)(aq^3-1-q-q^2)\right),\\
\mathsf{p}_1(x) =& \frac{(q-1)}{q^5}\left(aq^9x^2+q^4x(a^2q^4+a^2q^3-1-a-q-2aq-q^2-aq^2+aq^3+aq^4)\right.\\
&\left.+(aq^2-1)(aq^4-1-q-q^2+aq^3)\right),\\
\mathsf{p}_2(x) =& \frac{(q-1)^2}{q^6}\left(aq^8x^2(1+q+aq)+q^3x(a^2q^5+2a^2q^4-a-q-2aq-2aq^2+a^2q^3+aq^4)\right.\\
&\left.+(aq^2-1)(aq^3-1)\right),\\
\mathsf{p}_3(x) =& \frac{a(q-1)^3x}{q^4}\left(q^5x(1+a+aq)+(1+q)(-1+aq^3)\right),\\
\mathsf{p}_4(x) =& a^2(q-1)^4x^2.
\end{aligned}
$$

Proof. The proof follows after using an algorithm written in Wolfram Mathematica 13. □

4.6. The Stieltjes-Wigert polynomials

These polynomials can be written in terms of basic hypergeometric series as [14, §14.21]

$$S_n(x;q) = {}_1\phi_1\left(\begin{matrix} q^{-n} \\ 0 \end{matrix}; q, -q^{n+1}x\right). \tag{70}$$

In this case $\phi(x) = x^2$ and $\phi^*(x) = x$, i.e., $\mathsf{c} = 0$, and $S_n(\mathsf{c};q) = 1$. Taking this into account we can state the following result.

Lemma 10. *For any $n \in \mathbb{N}_0$, the following identities hold:*

$$
\begin{aligned}
\nabla_n S_n(x;q) &= -q^n x S_{n-1}(qx;q), &(71)\\
S_n(x;q) + q^{-n-1}\Delta_n S_n(x;q) &= S_{n+1}(x/q;q), &(72)\\
S_n(x;q) - (1-q^{-n})\nabla_n S_n(x;q) &= S_n(x/q;q). &(73)
\end{aligned}
$$

From Lemma 10 we can deduce some a new identities related to the Stieltjes-Wigert polynomials.

Theorem 12. *For any $n, k \in \mathbb{N}_0$, the following Rodrigues-type identities hold:*

$$
\begin{aligned}
S_n(x;q) &= \frac{1}{(-x)^k} q^{-n}\nabla_n \cdots q^{-n}\nabla_n S_{n+k}(x/q^k;q), &(74)\\
S_n(x;q) &= (-1)^n (q;q)_n q^{-n}\Delta_n \cdots q^{-n+k}\Delta_n \frac{(-1)^n}{(q;q)_{n-k}} S_{n-k}(xq^k;q). &(75)
\end{aligned}
$$

Proof. The first identity holds by mathematical induction on k after a straightforward simplification and using (71).

The second identity holds by mathematical induction on k after a straightforward simplification and using (72) written in the following way:

$$\frac{(-q)^n}{(q;q)_n} S_n(x;q) = \Delta_n \frac{(-1)^n}{(q;q)_{n-1}} S_{n-1}(xq;q).$$

□

The last result concerning the operators associated with the Stieltjes-Wigert polynomials is as follows.

	α_n	β_n	γ_n	d_n^2
L	$-n-\alpha-1$	$2n+\alpha+1$	$-n$	$\dfrac{n!\Gamma^2(\alpha+1)}{\Gamma(n+\alpha+1)}$
C	$-a$	$n+a$	$-n$	$\dfrac{n!}{a^n}$
M	$\dfrac{c(n+\beta)}{c-1}$	$-\dfrac{n+c(n+\beta)}{c-1}$	$\dfrac{n}{c-1}$	$\dfrac{n!\Gamma(\beta)}{\Gamma(\beta+n)c^n}$
M	$\dfrac{n+\beta}{c-1}$	$-\beta-\alpha_n-\gamma_n$	$\dfrac{nc}{c-1}$	$\dfrac{n!\Gamma(\beta)c^n}{\Gamma(\beta+n)}$
bqL	$(1-aq^{n+1})(1-bq^{n+1})$	$1-\alpha_n-\gamma_n$	$abq^{n+1}(q^n-1)$	$\dfrac{(q^{-1};q^{-1})_n\,q^n}{(a^{-1}q^{-1},b^{-1}q^{-1};q^{-1})_n}$
qM	$\dfrac{(c+q^{n+1})(bq^{n+1}-1)}{q^{2n+1}}$	$bq-\alpha_n-\gamma_n$	$\dfrac{cq(q^n-1)}{q^{2n+1}}$	$\dfrac{(q;q)_n}{(bq,-c^{-1}q;q)_n\,q^n}$
1qL	$q^n(aq^{n+1}-1)$	$-\alpha_n-\gamma_n$	$aq^n(q^n-1)$	$\dfrac{(q;q)_n a^n q^n}{(aq;q)_n}$
qL	$\dfrac{q^{n+1+\alpha}-1}{q^{2n+1+\alpha}}$	$-\alpha_n-\gamma_n$	$\dfrac{q(q^n-1)}{q^{2n+\alpha+1}}$	$\dfrac{(q;q)_n}{(q^{(\alpha+1)};q)q^n}$
qC	$-\dfrac{a+q^{n+1}}{q^{2n+1}}$	$-\alpha_n-\gamma_n$	$\dfrac{aq(q^n-1)}{q^{2n+1}}$	$\dfrac{(q;q)_n}{(-a^{-1}q,q)_n q^n}$
OLB	$-aq^{2n+1}$	$-\alpha_n-\gamma_n$	$-aq^n(q^n-1)$	$\dfrac{(q,q)_n a^n q^n}{(aq,q)_n}$
SW	$\dfrac{-1}{q^{2n+1}}$	$-\alpha_n-\gamma_n$	$\dfrac{q(q^n-1)}{q^{2n+1}}$	$\dfrac{(q;q)_n}{q^n}$

Table 1. Essential data of the Laguerre-type classical orthogonal polynomials

Proposition 5. *The following identity holds:*

$$(q-x)S_n(x;q)+(1-q)x\mathscr{D}_{1/q}S_n(x;q)=qS_{n+1}(x/q^2;q).$$

Proof. The proof follows after using an algorithm written in Wolfram Mathematica 13. □

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