On the factorization of the second order difference equation of hypergeometric type on non-uniform lattices

Roberto Costas-Santos

Orthogonal Polynomials and Approximation theory Group, Universidad de Sevilla, Spain

E-mail: roberto@euler.us.es

June 2002

The FM was already used by Darboux, Schrödinger, Infeld and Hull and others to obtain the solutions of certain differential equations. Later on, Miller extended it to difference and q-differences equations in the Hahn sense.
The classical FM was based on the existence of a so-called raising and lowering operators for the corresponding equation that allows to find the explicit solutions in a very easy way. Going further, Atakishiyev and coauthors have found the dynamical symmetry algebra related with the FM and the differential or difference equations [3]. Also Smirnov [8] have been worked in the similar work line. Using the idea by Bangerezako [4] for the Askey–Wilson polynomials and Lorente [6] solve the problem for the classical continuous and discrete cases. Here we will obtain the FM for the general polynomial solutions of the hypergeometric difference equation on the general quadratic nonuniform lattice $x(s) = c_1 q^s + c_2 q^{-s} + c_3$. We will use, as it is already suggested in [3, 8], not the polynomial solutions but the corresponding normalized functions which is more natural and useful. In such a way, the method proposed here is the generalization of [4] and [6] to the aforementioned nonuniform lattice.

1 The orthonormal functions on nonuniform lattices

Let $P_n(x(s))_q$ the polynomial solutions of the second order difference equation of hypergeometric type

$$\sigma(s)\frac{\Delta}{\Delta x(s-1/2)}\left[\frac{\nabla y(s)}{\nabla x(s)}\right]+\tau(s)\frac{\Delta y(s)}{\Delta x(s)}+\lambda y(s)=0. \qquad (1)$$

Here $\nabla f(s)=f(s)-f(s-1)$, $\Delta f(s)=f(s+1)-f(s)$, and $x(s)=C_1(q^s+q^{-s-\mu})+C_3$ being C_1, C_3 and μ constants.

Under certain conditions the polynomial solutions of (1) are orthogonal. For example, if $\sigma(s)\rho(s)x^k(s-1/2)\Big|_b^b=0, \quad \forall k\geq 0$, then the polynomials $P_n(s)_q$ satisfy the following discrete orthogonality property

$$\sum_{s=a}^{b-1}P_n(x(s))_qP_m(x(s))_q\rho(s)\Delta x(s-1/2)=\delta_{nm}d_n^2, \qquad (2)$$

where $\rho(s)$ is a solution of the Pearson-type equation

$$\frac{\triangle}{\Delta x(s-1/2)}\left[\sigma(s)\rho(s)\right]=\tau(s)\rho(s) \qquad (3)$$

From the orthogonality follows the Three Terms Recurrence Relation:

$$x(s)P_n(x(s))_q=\alpha_nP_{n+1}(x(s))_q+\beta_nP_n(x(s))_q+\gamma_nP_{n-1}(x(s))_q$$

where α_n, β_n and γ_n are constants.

Let us introduce the set of orthonormal functions which are orthogonal with respect to the unit weight

$$\varphi_n(s)=\sqrt{\rho(s)/d_n^2}P_n(x(s))_q, \qquad (4)$$

e.g. for the case of discrete orthogonality we have

$$\sum_{s_i=a}^{b-1}\varphi_n(s_i)\varphi_m(s_i)\Delta x(s_i-1/2)=\delta_{nm}.$$

2 Factorization of difference equation of hypergeometric type on the nonuniform lattice

We will define from (1) the following operator

$$H(s,n) \equiv \sqrt{\Theta(s-1)\sigma(s)}\frac{1}{\nabla x(s)}E^{-} + \sqrt{\Theta(s)\sigma(s+1)}\frac{1}{\Delta x(s)}E^{+} -$$
$$\left(\frac{\Theta(s)}{\Delta x(s)} + \frac{\sigma(s)}{\nabla x(s)} - \lambda_n \Delta x(s-1/2)\right) I,$$

where $E^{-}f(s) = f(s-1)$, $E^{+}f(s) = f(s+1)$ and $If(s) = f(s)$.

Clearly, the orthonormal functions satisfy $H(s,n)\varphi_n(s) = 0$, and the TTRR:

$$\alpha_n \frac{d_{n+1}}{d_n}\varphi_{n+1}(s) + \gamma_n \frac{d_{n-1}}{d_n}\varphi_{n-1}(s) + (\beta_n - x(s))\varphi_n(s) = 0, \tag{5}$$

Let us rewrite the raising and lowering operators in the following way

$$L^{+}(s,n) = u(s,n)I + \sqrt{\Theta(s-1)\sigma(s)}\frac{1}{\nabla x(s)}E^{-},$$
$$L^{-}(s,n) = v(s,n)I + \sqrt{\Theta(s)\sigma(s+1)}\frac{1}{\Delta x(s)}E^{+},$$

where, as before, $\Theta(s) = \sigma(s) + \tau(s)\Delta x(s-1/2)$, and

$$u(s,n) = \frac{\lambda_n}{[n]_q}\frac{\tau_n(s)}{\tau_n'} - \frac{\sigma(s)}{\nabla x(s)},$$
$$v(s,n) = -\frac{\lambda_n}{[n]_q}\frac{\tau_n(s)}{\tau_n'} + \lambda_n \Delta x(s-1/2) + \frac{\lambda_{2n}}{[2n]_q}(x(s) - \beta_n) - \frac{\Theta(s)}{\Delta x(s)}.$$

Proposition 2.1 *The operator H corresponding to the eigenvalue λ_n in the difference equation is self adjoint.*

Proposition 2.2 *The operators L^+ and L^- are mutually adjoint.*

$$\sum_{s_i=a}^{b-1} \varphi_{n+1}(s_i)\left[\frac{[2n]_q}{\lambda_{2n}} L^+(s_i,n)\varphi_n(s_i)\right]\Delta x(s_i-\tfrac{1}{2})$$
$$=\sum_{s_i=a}^{b-1}\left[\frac{[2n+2]_q}{\lambda_{2n+2}} L^-(s_i,n+1)\varphi_{n+1}(s_i)\right]\varphi_n(s_i)\Delta x(s_i-\tfrac{1}{2}) = \alpha_n \frac{d_{n+1}}{d_n},$$

and
$$L^+(s,n)\varphi_n(s) = \alpha_n \frac{\lambda_{2n}}{[2n]_q}\frac{d_{n+1}}{d_n}\varphi_{n+1}(s)$$
$$L^-(s,n)\varphi_n(s) = \gamma_n \frac{\lambda_{2n}}{[2n]_q}\frac{d_{n-1}}{d_n}\varphi_{n-1}(s).$$

Furthermore
$$L^-(s,n+1)L^+(s,n) = h^{\mp}(n)I + u(s+1,n)H(s,n),$$
$$L^+(s,n-1)L^-(s,n) = h^{\pm}(n)I + u(s,n-1)H(s,n),$$

where
$$h^{\pm}(n) = \frac{\lambda_{2n-2}}{[2n-2]_q}\frac{\lambda_{2n}}{[2n]_q}\alpha_{n-1}\gamma_n.$$

All the above results lead us to our main theorem:

Theorem 2.1 *The operator $H(s,n)$, corresponding to the hypergeometric difference equation for orthonormal functions $\varphi_n(s)$, admits the following factorization –usually called the Infeld-Hull-type factorization–*

$$u(s+1,n)H(s,n) = L^-(s,n+1)L^+(s,n) - h^{\mp}(n)I, \qquad (6)$$

$$u(s,n)H(s,n+1) = L^+(s,n)L^-(s,n+1) - h^{\mp}(n)I, \qquad (7)$$

respectively.

Remark: Substituting in the above formulas the expression $x(s)=s$ we obtain the corresponding results for the uniform lattice cases (Hahn, Kravchuk, Meixner and Charlier), considered before by Atakishiyev, Lorente, Smirnov, ... and by taking appropriate limits we can recover the classical continuous case (Jacobi, Laguerre and Hermite).

3 Applications to q-normalized orthogonal functions

For the sake of completeness we will apply the above results to several families of orthogonal q-polynomials and their corresponding orthonormal q-functions that are of interest and appear in several branches of mathematical physics.

The big q-Jacobi functions

Now we will consider the most general family of q-polynomials on the exponential lattice, the big q-Jacobi polynomials, that appear in the representation theory of the quantum algebras. They were introduced by Hahn in 1949 and are defined by

$$P_n(x;a,b,c;q) = \frac{(aq;q)_n(cq;q)_n}{(abq^{n+1};q)_n}\, {}_3\phi_2\left(\begin{array}{c} q^{-n}, abq^{n+1}, x \\ aq, cq \end{array} \middle| \, q;q \right), \quad x \equiv q^s$$

The corresponding Hamiltonian is

$$\begin{aligned} H(x,n) = {} & \frac{\sqrt{a(x-q)(x-aq)(x-cq)(bx-cq)}}{x(q-1)} E^- + \\ & q\frac{\sqrt{a(x-1)(x-a)(x-c)(bx-c)}}{x(q-1)} E^+ + \\ & \left(\frac{1+abq^{2n+1}}{q^n(1-q)} x - \frac{q(a+ab+c+ac)}{1-q} + \frac{acq(q+1)}{1-q} x^{-1} \right) I. \end{aligned}$$

In this case

$$L^-(x,n+1)L^+(x,n) = \delta_{n+1}\gamma_{n+1} I + v(x,n+1)H(x,n),$$

$$L^+(x,n-1)L^-(x,n) = \delta_n\gamma_n I + u(x,n-1)H(x,n),$$

being

$$\delta_n = \frac{(1-abq^{2n-1})(1-abq^{2n+1})}{q^{2n-1}(q-1)^2}.$$

The above formulas are the factorization formulas for the family of the big q-Jacobi normalized functions.
Since all discrete q-polynomials on the exponential lattice $x(s) = c_1 q^s + c_3$ —the so called, q-Hahn class— can be obtained from the big q-Jacobi polynomials by a certain limit process, then from the above formulas we can obtain the factorization formulas for the all other cases in the q-Hahn tableau. Of special interest are the q-Hahn polynomials and the big q-Laguerre polynomials, which are particular cases of the big q-Jacobi polynomials when $c = q^{-N-1}$, $N = 1, 2, \ldots$, and $c = 0$, respectively.

3.1 The Askey–Wilson functions

Finally we will consider the family of Askey–Wilson polynomials. They are polynomials on the lattice $x(s) = 1/2(q^s + q^{-s}) \equiv x$, defined by

$$p_n(x(s); a, b, c, d) = \frac{(ab;q)_n(ac;q)_n(ad;q)_n}{a^n} \times$$

$${}_4\phi_3\left(\begin{array}{c} q^{-n}, q^{n-1}abcd, ae^{-i\theta}, ae^{i\theta} \\ ab, ac, ad \end{array}\middle| q; q\right),$$

that correspond to the general case $q^{s_1} = a$, $q^{s_2} = b$, $q^{s_3} = c$, $q^{s_4} = d$.

Their orthogonality relation is of the form

$$\int_{-1}^{1} \omega(x) p_n(x; a, b, c, d) p_m(x; a, b, c, d) \sqrt{1-x^2}\kappa_q dx = \delta_{nm} d_n^2, \quad x = \cos\theta,$$

where

$$\omega(x) = \frac{h(x,1)h(x,-1)h(x,q^{\frac{1}{2}})h(x,-q^{\frac{1}{2}})}{2\pi\kappa_q(1-x^2)h(x,a)h(x,b)h(x,c)h(x,d)},$$

$$h(x,\alpha) = \prod_{k=0}^{\infty}[1 - 2\alpha x q^k + \alpha^2 q^{2k}],$$

Defining now the normalized functions $\sqrt{\omega(x)/d_n^2} p_n(x;a,b,c,d)$,

the corresponding Hamiltonian $H(s,n)$ is

$$\begin{aligned} H(s,n) = \; & \frac{2q^{3/2}}{[2s-1]_q} G(s,a,b,c,d)E^- + \frac{2q^{3/2}}{[2s+1]_q} G(s+1,a,b,c,d)E^+ + \\ & 2\Big(q^{-2s+1/2}\frac{\prod_{i=1}^4(1-q^{s_i+s})}{[2s+1]_q} + q^{-2s+1/2}\frac{\prod_{i=1}^4(q^s-q^{s_i})}{[2s-1]_q} + \\ & q^{-n+1}\kappa_q^2(1-q^n)(1-abcdq^{n-1})[2s]_q\Big) I \end{aligned}$$

where

$$G(s,a,b,c,d) = \sqrt{\prod_{i=1}^{4}(1 - 2q^{s_i}q^{-1/2}x(s-1/2) + q^{-1}q^{2s_i})},$$

Thus,

$$L^-(s,n+1)L^+(s,n) = D_{2n}D_{2n+2}\gamma_{n+1}I + v(s,n+1)H(s,n),$$

and

$$L^+(s,n-1)L^-(s,n) = D_{2n-2}D_{2n}\gamma_n I + u(s,n-1)H(s,n),$$

where

$$D_n = -4q^{-n/2+1/2}(q-1)(1-abcdq^{n-1}).$$

which is the factorization formula for the Askey–Wilson functions. Finally, let us consider the special case of Askey–Wilson polynomials when $a=b=c=d=0$, i.e., the continuous q-Hermite polynomials which are closely related with the q-harmonic oscillator model introduced in 1989 by Biedenharn and Macfarlane. The factorization for this "simple" case was considered by Atakishiyev and Suslov in 1991.

Obviously the factorization for this case follows from the Askey-Wilson case substituting $a=b=c=d=0$ in the above formulas.

What next?

Problem 1:

> To find two operators $a(s)$ and $b(s)$ and a constant ς such that the Hamiltonian $H_q(s) = b(s)a(s)$ and $[a(s), b(s)]_\varsigma := a(s)b(s) - \varsigma b(s)a(s) = I$.

There are previous works by Atakishiyev and Suslov: $q-$Hermite continuous, Atakishiev, Suslov and Askey: q-Charlier, Suslov and Askey: Al-Salam y Chihara

> **Definition:** *A function $f(z)$ is said to be a q-linear function of z if there exist two functions F and G independent of z such that for all $z, \zeta \in \mathbb{C}$*
>
> $$f(z+\zeta) = F(\zeta)f(z) + G(\zeta).$$

Important result: The solution of the problem 1 exists iff λ_n is a q-linear or q^{-1}-linear function of n.

All the known q-examples: satisfy the above condition.

Open questions:

1) When the operators are mutually adjoint?

2) Applications to some q-models of the linear oscillator

References

[1] R. Álvarez-Nodarse and R.S. Costas-Santos, On the factorization of the difference equation of hypergeometric-type in nonuniform lattices. *J. Phys. A: Math. Gen.* **34** (2001), 5551-5569.

[2] N. M. Atakishiyev and S. K. Suslov, A realization of the q-harmonic oscillator. *Theoret. and Math. Phys.* **87** (1991), 1055-1062.

[3] N. M. Atakishiyev, A. Frank, and K. B. Wolf, A simple difference realization of the Heisenberg q-algebra. *J. Math. Phys.* **35** (1994), 3253-3260.

[4] G. Bangerezako, The factorization method for the Askey–Wilson polynomials. *J. Comput. Appl. Math.* **107** (1999), 219-232.

[5] L. Infeld and T. E. Hull, The factorization method. *Rev. Modern Physics* **23** (1951), 21–68.

[6] M. Lorente, Raising and lowering operators, factorization method and differential/difference operators of hypergeometric type. *J. Phys. A: Math. Gen.* **34** (2001), 569–568.

[7] A. F. Nikiforov, S. K. Suslov, and V. B. Uvarov, *Classical Orthogonal Polynomials of a Discrete Variable.* Springer Series in Computational Physics. Springer-Verlag, Berlin, 1991.

[8] Yu. F. Smirnov, On factorization and algebraization of difference equation of hypergeometric type. In *Proc. Internat. Workshop on Orthogonal Polynomials in Mathematical Physics*, M. Alfaro et al. Eds. Servicio de Publicaciones de la Universidad Carlos III de Madrid, (1997) 153-161.

[9] V. Spiridonov, L. Vinet, A. Zhedanov, Difference Schrödinger operators with linear and exponential discrete spectra. *Lett. Math. Phys.* **29** (1993), 63–73.