Recent results on Semiclassical Orthogonal Polynomials

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Classical orthogonal polynomials

Given a linear functional $\mathbf{u}:\mathbb{P}[x]\to\mathbb{C}$ fulfilling the Pearson type equation

(PE) $\mathscr{D}\big(\phi(x)\mathbf{u}\big)=\psi(x)\mathbf{u},\quad \deg\phi\le 2, \deg\psi=1 \quad \to p_n(x)=\text{ops}(\mathbf{u})$

(TTRR) $xp_n(x)=\alpha_n p_{n+1}(x)+\beta_n p_n(x)+\gamma_n p_{n-1}(x) \quad \gamma_n=\alpha_{n-1}\frac{\langle \mathbf{u},p_n^2(x)\rangle}{\langle \mathbf{u},p_{n-1}^2(x)\rangle}$

(1RE) $\phi(x)(\mathscr{D}p_n)(x)=\widetilde{\alpha}_n p_{n+1}(x)+\widetilde{\beta}_n p_n(x)+\widetilde{\gamma}_n p_{n-1}(x) \quad \widetilde{\gamma}_n\ne 0$

(2RE) $p_n(x)=\widehat{\alpha}_n(\mathscr{D}p_{n+1})(x)+\widehat{\beta}_n(\mathscr{D}p_n)(x)+\widehat{\gamma}_n(\mathscr{D}p_{n-1})(x) \quad \widehat{\gamma}_n\ne 0,\ \widehat{\gamma}_n\ne\gamma_n$

(SODE) $\phi(x)\mathscr{D}\mathscr{D}^* p_n(x)+\psi(x)\mathscr{D}p_n(x)+\lambda_n p_n(x)=0 \quad \lambda_n=-n\psi'-\frac{n(n-1)}{2}\phi''(x)$

where $\mathscr{D}:=\frac{d}{dx}$ in the cont. case, $\mathscr{D}:=\Delta$ in the disc. case, $\mathscr{D}:=\mathscr{D}_q$ in the q-disc. case.

Some of my results

Degenerate Favard's Theorem [2, 4]

If there exists a unique $N\in\mathbb{N}$ with $\gamma_N=0$, then (p_n) is an OPS with respect to the bilinear functional ($\mathscr{D}$-Sobolev orthogonality)

$$\langle f,g\rangle_S=\langle \mathbf{u},fg\rangle+\langle \mathbf{v},(\mathscr{D}^N f)(x)(\mathscr{D}^N g)(x)\rangle.$$

where $\mathbf{v}$ is certain linear functional related to $\mathbf{u}$ and the integer value N; and p_n can be factorized as follows: $p_{N+m}(x)=p_N(x)q_m(x)$.

Extension of the Hahn's Theorem [3]

Let $\mathbf{u}$ be a q-classical linear functional, and let p_n be an ops($\mathbf{u}$) (q-quadratic lattice, i.e. $x(s)=c_1q^s+c_2q^{-s}+c_3$) then:

$$(\mathscr{D}_q p_{n+1})(x)\text{ is orthogonal}\iff \mathbf{u}\text{ is }q\text{-classical}.$$

Semiclassical orthogonal polynomials

Given a linear functional $\mathbf{u}:\mathbb{P}[x]\to\mathbb{C}$ fulfilling the Pearson type equation

(PE) $\mathscr{D}\big(\phi(x)\mathbf{u}\big)=\psi(x)\mathbf{u},\quad \phi,\psi\in\mathbb{P}[x] \quad \to p_n(x)=\text{qops}(\mathbf{u})$

(QO-r) $\langle \mathbf{u},p_n(x)p_m(x)\rangle=0,\quad |n-m|>r,$ and $\exists M\ge r$ so that $\langle \mathbf{u},p_M(x)p_{M-r}(x)\rangle\ne 0$

(Class) $\sigma_{\mathbf{u}}$: Given the set $\mathbb{P}^2_{\mathbf{u}}:=\{(\phi,\psi)\in\mathbb{P}^2\ :\ \mathscr{D}\big(\phi(x)\mathbf{u}\big)=\psi(x)\mathbf{u}\}$, the class of $\mathbf{u}$ is

$$\sigma_{\mathbf{u}}=\min_{(\phi,\psi)\in\mathbb{P}^2_{\mathbf{u}}}\max\{\deg\phi-2,\deg\psi-1\}.$$

(RR) There exists $\mathscr{M}(x)\in\mathbb{P}[x]$ such that $\mathscr{M}(x)p_n(x)=\sum_{\nu=-m}^{m}\theta_{n,\nu}p_\mu(x)$ [7]

(1RE) $\mathscr{D}p_n(x)=\mathscr{K}_n(x;\phi(x))$ [6] $\mathscr{K}_n(x;\phi(x))$ is a generalized kernel.

(2RE) $\sum_{\nu=n-\sigma_{\mathbf{u}}}^{n+\deg\phi}\alpha_{n,\nu}p_\nu(x)=\sum_{\nu=n-\deg\phi}^{n+\deg\phi}\beta_{n,\nu}(\mathscr{D}p_{\nu+1})(x)$ [1]

Some of my results

Characterization of semiclassical quasi-orthogonal polynomials [6]

Given $\phi(x)\in\mathbb{P}[x]$, with $\phi(x)=(x-c)\widetilde{\phi}(x)$, then

$$\mathscr{K}_n(x;\phi(x))=\frac{K_{n+1}(x;\widetilde{\phi}(x))K_n(c;\widetilde{\phi}(x))-K_{n+1}(c;\widetilde{\phi}(x))K_n(x;\widetilde{\phi}(x))}{x-c}.$$

Recurrence relations for semiclassical orthogonal polynomials [7]

Examples $p(x)\mathbf{v}=r(x)\mathbf{u},\quad p(x),r(x)\in\mathbb{P}[x]$

Transformations:

- Christoffel transform: $\mathbf{v}=(x-c)\mathbf{u}$.
- Uvarov transform: $\mathbf{v}=\mathbf{u}+M\delta_{x-c}$, $M>0 \longrightarrow (x-c)\mathbf{v}=(x-c)\mathbf{u}$.
- Geronimus transform: $\mathbf{v}=(x-c)^{-1}\mathbf{u}+M\delta_{x-c}$, $M>0 \longrightarrow (x-c)\mathbf{v}=\mathbf{u}$.

In recent works, the Exceptional Polynomials (EP) are orthogonal with respect to the weight $\omega(x)/p_M^2(x)$, where $\omega(x)$ is a classical weight (Jacobi, Laguerre, Hermite, etc.).

The Krall Polynomials are also semiclassical polynomials, orthogonal with respect to $\mathbf{v}$ so that $p(x)\mathbf{v}=\mathbf{u}$, where $p(x)$ could have zeros with algebraic multiplicity >1.

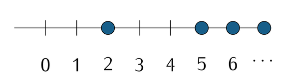
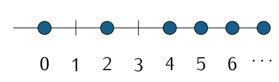
Some of my results

Analytic properties of some basic hypergeometric-Sobolev type orthog. polynom. [5]

Three papers about Exceptional Polynomials with Kelly, J. S.:

- *About Laguerre Exceptional Polynomials.*

  Abstract: *In this contribution we obtain algebraic, analytic and spectral properties of the Laguerre EP where the flags are*

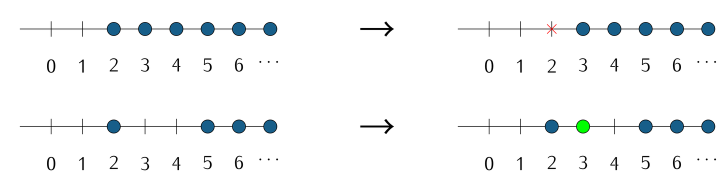
   *or* 

- *Exceptional orthogonal polynomials from the semi-classical point of view.*

  Abstract: *In this contribution we extend part of the the theory of EP, we also consider a new notation for these polynomials presenting new expressions for the operators involved in the problem. We also explain most of the analytic properties related to these polynomials from the semiclassical point of view.*

- *Algebraic and analytic properties of Exceptional Polynomials.*

  Abstract: *In this contribution we study the concept of stretching/compressing the flag, and how such changes affect to the operators involved in the process of construction of these polynomials.*



stretching (OFF)

compressing (ON)

J. Dini and P. Maroni (1990) got the following identity: Given a polynomial $R(x)$ with distinct roots $r_1,\dots,r_m$,

$$\langle R^{-1}(x)\mathbf{u},p\rangle=\left\langle \mathbf{u},\frac{p(x)-\sum_{i=1}^m p(r_i)L_i(x)}{R(x)}\right\rangle,\ \text{where } L_i(x)=\frac{R(x)}{R'(x_i)(x-r_i)},\quad i=1,\dots,m.$$

I have extended this result when $R(x)$ has zeros with multiplicities >1 and apply such result to the precious problems as well as to other related with Gaussian quadrature.

Further problems I am interested in:

Differentia/difference equation associated to Semiclassical OP, singular semiclasical functionals, connection Krall-Exceptionals, among others.

References

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[7] Costas-Santos, R. S. *Recurrence relations for semiclassical orthogonal polynomials*, 2018. In progress.