# Basic hypergeometric transformations from symmetric and $q$-inverse sub-families of the Askey-Wilson polynomials in the $q$-Askey scheme

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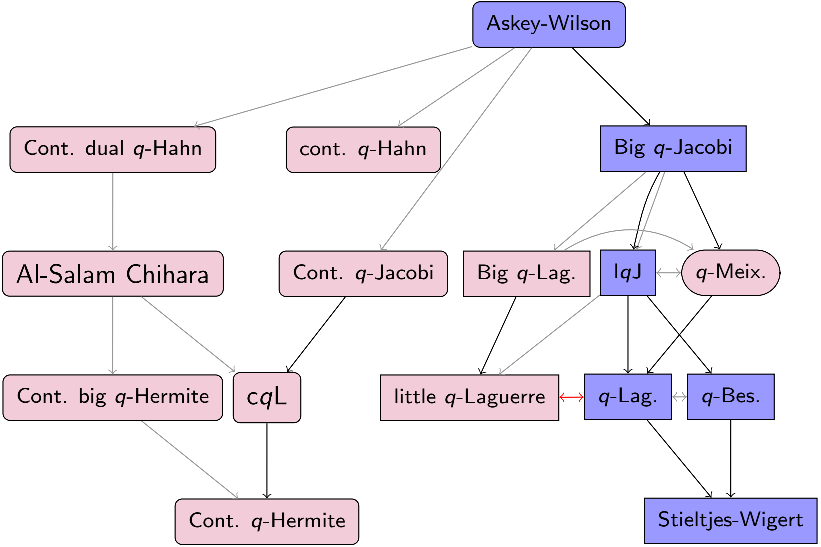
# Outline

# The basic hypergeometric scheme



The basic hypergeometric series, which we will often use, is defined for $q, z \in \mathbb{C}^\star$ such that $|q| < 1$, $s, r \in \mathbb{N}_0$, $b_j \notin \Omega_q$, $j = 1, ..., s$, as

$$ {}_r\phi_s\left(\begin{matrix} a_1, ..., a_r \\ b_1, ..., b_s \end{matrix}; q, z\right) = \sum_{k=0}^{\infty} \frac{(a_1; q)_k \cdots (a_r; q)_k}{(q; q)_k (b_1; q)_k \cdots (b_s; q)_k} \left((-1)^k q^{\binom{k}{2}}\right)^{1+s-r} z^k, $$

where

$$ \Omega_q := \{q^{-n} : n \in \mathbb{N}\}, $$

$$ (a; q)_n := (1 - a)(1 - aq) \cdots (1 - aq^{n-1}), \quad n \in \mathbb{N}_0, $$

and

$$ (a_1, \ldots, a_l; q)_n := (a_1; q)_n \cdots (a_l; q)_n. $$

# Limit transition formulas

$$\lim_{\lambda\to\infty} {}_r\phi_s\left(\begin{matrix} a_1, \dots, a_{r-1}, \lambda a_r \\ b_1, \dots, b_s \end{matrix}; q, \frac{z}{\lambda}\right) = {}_{r-1}\phi_s\left(\begin{matrix} a_1, \dots, a_{r-1} \\ b_1, \dots, b_s \end{matrix}; q, z a_r\right),$$

$$\lim_{\lambda\to\infty} {}_r\phi_s\left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_{s-1}, \lambda b_s \end{matrix}; q, \lambda z\right) = {}_r\phi_{s-1}\left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_{s-1} \end{matrix}; q, \frac{z}{b_s}\right),$$

$$\lim_{\lambda\to\infty} {}_r\phi_s\left(\begin{matrix} a_1, \dots, a_{r-1}, \lambda a_r \\ b_1, \dots, b_{s-1}, \lambda b_s \end{matrix}; q, z\right) = {}_{r-1}\phi_{s-1}\left(\begin{matrix} a_1, \dots, a_{r-1} \\ b_1, \dots, b_{s-1} \end{matrix}; q, z\frac{a_r}{b_s}\right).$$

**Theorem:** [Gasper & Rahman (1990)]

Let $m, n, k, r, s \in \mathbb{N}_0$, $0 \leq k \leq r$, $0 \leq m \leq s$, $a_k \in \mathbb{C}$, $b_m \in \Omega_q$, $q \in \mathbb{C}^*$ such that $|q| \neq 1$. Then,

$$
{}_{r+1}\phi_s\left(\begin{matrix} q^{-n}, a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix}; q, z\right) = \frac{(a_1, \ldots, a_r; q)_n}{(b_1, \ldots, b_s; q)_n} \left(\frac{z}{q}\right)^n \left((-1)^n q^{\binom{n}{2}}\right)^{s-r-1}
$$

$$
\times \sum_{k=0}^{n} \frac{\left(q^{-n}, \frac{q^{1-n}}{b_1}, \ldots, \frac{q^{1-n}}{b_s}; q\right)_k}{\left(q, \frac{q^{1-n}}{a_1}, \ldots, \frac{q^{1-n}}{a_r}; q\right)_k} \left(\frac{b_1 \cdots b_s}{a_1 \cdots a_r} \frac{q^{n+1}}{z}\right)^k .
$$

Let $p \in \mathbb{N}_0$, $s \geq r \geq p \geq 0$. Then one has

$$
{}_{r+1}\phi_s\left(\begin{matrix} q^{-n}, a_1, \ldots, a_{r-p}, \overbrace{0, \ldots, 0}^{p} \\ b_1, \ldots, b_s \end{matrix}; q, z\right) = \frac{(a_1, \ldots, a_{r-p}; q)_n}{(b_1, \ldots, b_s; q)_n} \left(\frac{z}{q}\right)^n \left((-1)^n q^{\binom{n}{2}}\right)^{s-r-1}
$$

$$
\times\, {}_{s+1}\phi_s\left(\begin{matrix} q^{-n}, \frac{q^{1-n}}{b_1}, \ldots, \frac{q^{1-n}}{b_s} \\ \frac{q^{1-n}}{a_1}, \ldots, \frac{q^{1-n}}{a_{r-p}}, \underbrace{0, \ldots, 0}_{s-r} \end{matrix}; q, \frac{b_1 \cdots b_s}{a_1 \cdots a_{r-p}} \frac{q^{(p+1)n+1-p}}{z}\right).
$$

Let $r \geq s$, $r \geq p \geq 0$. Then one has

$$
{}_{r+1}\phi_s\left(\begin{matrix} q^{-n}, a_1, \ldots, a_{r-p}, \overbrace{0, \ldots, 0}^{p} \\ b_1, \ldots, b_s \end{matrix}; q, z\right) = \frac{(a_1, \ldots, a_{r-p}; q)_n}{(b_1, \ldots, b_s; q)_n} \left(\frac{z}{q}\right)^n \left((-1)^n q^{\binom{n}{2}}\right)^{s-r-1}
$$

$$
\times\, {}_{r+1}\phi_{r-p}\left(\begin{matrix} q^{-n}, \frac{q^{1-n}}{b_1}, \ldots, \frac{q^{1-n}}{b_s}, \overbrace{0, \ldots, 0}^{r-s} \\ \frac{q^{1-n}}{a_1}, \ldots, \frac{q^{1-n}}{a_{r-p}} \end{matrix}; q, \frac{b_1 \cdots b_s}{a_1 \cdots a_{r-p}} \frac{q^{(p+1)n+1-p}}{z}\right).
$$

Let $r \geq s \geq p \geq 0$. Then one has

$$
{}_{r+1}\phi_s\left(\begin{matrix} q^{-n}, a_1, \ldots, a_r \\ b_1, \ldots, b_{s-p}, \underbrace{0, \ldots, 0}_{p} \end{matrix}; q, z\right) = \frac{(a_1, \ldots, a_r; q)_n}{(b_1, \ldots, b_{s-p}; q)_n} \left(\frac{z}{q}\right)^n \left((-1)^n q^{\binom{n}{2}}\right)^{s-r-1}
$$

$$
\times\, {}_{r-p+1}\phi_r\left(\begin{matrix} q^{-n}, \frac{q^{1-n}}{b_1}, \ldots, \frac{q^{1-n}}{b_{s-p}}, \overbrace{0, \ldots, 0}^{r-s} \\ \frac{q^{1-n}}{a_1}, \ldots, \frac{q^{1-n}}{a_r} \end{matrix}; q, \frac{b_1 \cdots b_{s-p}}{a_1 \cdots a_r} \frac{q^{(1-p)n+p+1}}{z}\right).
$$

Let $s \geq r$, $s \geq p \geq 0$. Then one has

$$
{}_{r+1}\phi_s\left(\begin{matrix} q^{-n}, a_1, \ldots, a_r \\ b_1, \ldots, b_{s-p}, \underbrace{0, \ldots, 0}_{p} \end{matrix}; q, z\right) = \frac{(a_1, \ldots, a_r; q)_n}{(b_1, \ldots, b_{s-p}; q)_n} \left(\frac{z}{q}\right)^n \left((-1)^n q^{\binom{n}{2}}\right)^{s-r-1}
$$

$$
\times\, {}_{s-p+1}\phi_s\left(\begin{matrix} q^{-n}, \frac{q^{1-n}}{b_1}, \ldots, \frac{q^{1-n}}{b_{s-p}} \\ \frac{q^{1-n}}{a_1}, \ldots, \frac{q^{1-n}}{a_r}, \underbrace{0, \ldots, 0}_{s-r} \end{matrix}; q, \frac{b_1 \cdots b_{s-p}}{a_1 \cdots a_r} \frac{q^{(1-p)n+p+1}}{z}\right).
$$

# The Askey-Wilson polynomials

Define the sets $\mathbf{4} := \{1,2,3,4\}$, $\mathbf{a} := (a_1, a_2, a_3, a_4)$, $a_k \in \mathbb{C}^*$, $k \in \mathbf{4}$, and $x = \cos\theta \in [-1,1]$.

Let $n \in \mathbb{N}_0$, $p, s, r, t, u \in \mathbf{4}$, $p, r, t, u$ distinct and fixed,

$$p_n(x;\mathbf{a}|q) = a_p^{-n} \left(\{a_p a_s\}_{s\neq p}; q\right)_n \, {}_4\phi_3\left(\begin{matrix} q^{-n}, q^{n-1}a_{1234}, a_p e^{\pm i\theta} \\ \{a_p a_s\}_{s\neq p} \end{matrix}; q, q\right)$$

$$p_n(x;\mathbf{a}|q) = e^{in\theta}\left(a_{pr}, a_t e^{-i\theta}, a_u e^{-i\theta}; q\right)_n \, {}_4\phi_3\left(\begin{matrix} q^{-n}, a_p e^{i\theta}, a_r e^{i\theta}, \frac{q^{1-n}}{a_{tu}} \\ a_{pr}, \frac{q^{1-n}e^{i\theta}}{a_t}, \frac{q^{1-n}e^{i\theta}}{a_u} \end{matrix}; q, q\right)$$

where $a_{ij} := a_i a_j$, $a_{1234} = a_1 a_2 a_3 a_4$, $ae^{\pm i\theta} = \{ae^{i\theta}, ae^{-i\theta}\}$, $\{gf_s\} := g\{f_1, f_2, f_3, f_4\}$, and $\{gf_s\}_{s\neq p} := \{gf_s\} \setminus \{gf_p\}$.

**Consequence:** Let $n \in \mathbb{N}_0$, $b, c, d, e, f, q \in \mathbb{C}^*$, such that $|q| \neq 1$. Then, one has the following transformation formula for a terminating ${}_4\phi_3$:

$${}_4\phi_3\left(\begin{matrix} q^{-n}, c, d, \frac{qb}{ef} \\ q^{-n}\frac{cd}{b}, \frac{qb}{e}, \frac{qb}{f} \end{matrix}; q, q\right) = \frac{(e, f; q)_n}{\left(\frac{q^{-n}e}{b}, \frac{q^{-n}f}{b}; q\right)_n (q^n b)^n} \, {}_4\phi_3\left(\begin{matrix} q^{-n}, \frac{q^{-n}c}{b}, \frac{q^{-n}d}{b}, \frac{qb}{ef} \\ q^{-n}\frac{cd}{b}, \frac{q^{1-n}}{e}, \frac{q^{1-n}}{f} \end{matrix}; q, q\right).$$

# The Askey-Wilson polynomials

$$p_n(x;\mathbf{a}|q) = q^{-\binom{n}{2}}(-a_p)^{-n}\left(q^{n-1}a_{1234}, a_p e^{\pm i\theta};q\right)_n {}_4\phi_3\left(\begin{matrix} q^{-n}, \left\{\frac{q^{1-n}}{a_{ps}}\right\}_{s\neq p} \\ \frac{q^{2-2n}}{a_{1234}}, \frac{q^{1-n}e^{\pm i\theta}}{a_p} \end{matrix}; q, q\right)$$

$$p_n(x;\mathbf{a}|q) = \frac{\left(a_{pt}, a_{pu}, a_r e^{\pm i\theta};q\right)_n}{a_p^n\left(\frac{a_r}{a_p};q\right)_n} {}_8\phi_7\left(\begin{matrix} q^{-n}, \frac{q^{-n}a_p}{a_r}, \pm q^{1-\frac{n}{2}}\left(\frac{a_p}{a_r}\right)^{\frac{1}{2}}, \frac{q^{1-n}}{a_{rt}}, \frac{q^{1-n}}{a_{ru}}, a_p e^{\pm i\theta} \\ \pm q^{-\frac{n}{2}}\left(\frac{a_p}{a_r}\right)^{\frac{1}{2}}, a_{pt}, a_{pu}, \frac{q a_p}{a_r}, \frac{q^{1-n}e^{\pm i\theta}}{a_r} \end{matrix}; q, q^n a_{tu}\right)$$

$$p_n(x;\mathbf{a}|q) = e^{in\theta}\frac{\left(\{a_s e^{-i\theta}\};q\right)_n}{\left(e^{-2i\theta};q\right)_n} {}_8\phi_7\left(\begin{matrix} q^{-n}, q^{-n}e^{2i\theta}, \pm q^{1-\frac{n}{2}}e^{i\theta}, \{a_s e^{i\theta}\} \\ \pm q^{-\frac{n}{2}}e^{i\theta}, q e^{2i\theta}, \left\{\frac{q^{1-n}e^{i\theta}}{a_s}\right\} \end{matrix}; q, \frac{q^{2-n}}{a_{1234}}\right).$$

**Consequence:** Let $n \in \mathbb{N}_0$, $b, c, d, e, f, q \in \mathbb{C}^*$, such that $|q| \neq 1$. Then, one has the following transformation formula for a terminating ${}_8\phi_7$:

$$ {}_8\phi_7\left(\begin{matrix} q^{-n}, b, \pm q\sqrt{b}, c, d, e, f \\ \pm\sqrt{b}, q^{n+1}b, \frac{qb}{c}, \frac{qb}{d}, \frac{qb}{e}, \frac{qb}{f} \end{matrix}; q, \frac{q^{n+2}b^2}{cdef}\right) = \frac{\left(q^{-n}b^{-1}, c, d, e, f;q\right)_n}{\left(\frac{q^{-n}c}{b}, \frac{q^{-n}d}{b}, \frac{q^{-n}e}{b}, \frac{q^{-n}f}{b}, q^n b;q\right)_n (q^n b)^n}$$

$$\times {}_8\phi_7\left(\begin{matrix} q^{-n}, \frac{q^{-2n}}{b}, \pm\frac{q^{1-n}}{\sqrt{b}}, \frac{q^{-n}c}{b}, \frac{q^{-n}d}{b}, \frac{q^{-n}e}{b}, \frac{q^{-n}f}{b} \\ \pm\frac{q^{-n}}{\sqrt{b}}, \frac{q^{1-n}}{b}, \frac{q^{1-n}}{c}, \frac{q^{1-n}}{d}, \frac{q^{1-n}}{e}, \frac{q^{1-n}}{f} \end{matrix}; q, \frac{q^{n+2}b^2}{cdef}\right).$$

# Identity for BHS to go from $q$ to $1/q$

One interesting equality we can use is the following connecting relation between basic hypergeometric series on $q$, and on $q^{-1}$:

$$
{}_{r+1}\phi_r\left(\begin{matrix} q^{-n}, a_1, \ldots, a_r \\ b_1, \ldots, b_r \end{matrix}; q, z\right) = {}_{r+1}\phi_r\left(\begin{matrix} q^{n}, a_1^{-1}, \ldots, a_r^{-1} \\ b_1^{-1}, \ldots, b_r^{-1} \end{matrix}; q^{-1}, \frac{z a_1 a_2 \cdots a_r}{b_1 b_2 \cdots b_r}\right)
$$

$$
= \frac{(a_1, \ldots, a_r; q)_n}{(b_1, \ldots, b_r; q)_n} \left(-\frac{z}{q}\right)^n q^{-\binom{n}{2}} {}_{r+1}\phi_r\left(\begin{matrix} q^{-n}, \frac{q^{1-n}}{b_1}, \ldots, \frac{q^{1-n}}{b_r} \\ \frac{q^{1-n}}{a_1}, \ldots, \frac{q^{1-n}}{a_r} \end{matrix}; q, \frac{b_1 \cdots b_r}{a_1 \cdots a_r} \frac{q^{n+1}}{z}\right).
$$

# The $q$-inverse Askey Wilson polynomials

$$p_n(x;\mathbf{a}|q^{-1}) = \frac{(-a_p a_{1234})^n}{q^{3\binom{n}{2}}} \left( \left\{ \frac{1}{a_p a_s} \right\}_{s\neq p}; q \right)_n {}_4\phi_3 \left( \begin{matrix} q^{-n}, \frac{q^{n-1}}{a_{1234}}, \frac{e^{\pm i\theta}}{a_p} \\ \left\{ \frac{1}{a_p a_s} \right\}_{s\neq p} \end{matrix} ; q, q \right)$$

$$p_n(x;\mathbf{a}|q^{-1}) = \frac{\left(-a_{1234} e^{-i\theta}\right)^n}{q^{3\binom{n}{2}}} \left( \frac{1}{a_{pr}}, \frac{e^{i\theta}}{a_t}, \frac{e^{i\theta}}{a_u}; q \right)_n {}_4\phi_3 \left( \begin{matrix} q^{-n}, \frac{e^{-i\theta}}{a_p}, \frac{e^{-i\theta}}{a_r}, q^{1-n} a_{tu} \\ \frac{1}{a_{pr}}, q^{1-n} a_t e^{-i\theta}, q^{1-n} a_u e^{-i\theta} \end{matrix} ; q, q \right)$$

$$p_n(x;\mathbf{a}|q^{-1}) = \frac{(a_p a_{1234})^n}{q^{4\binom{n}{2}}} \left( \frac{q^{n-1}}{a_{1234}}, \frac{e^{\pm i\theta}}{a_p}; q \right)_n {}_4\phi_3 \left( \begin{matrix} q^{-n}, \{q^{1-n} a_{ps}\}_{s\neq p} \\ q^{2-2n} a_{1234}, q^{1-n} a_p e^{\pm i\theta} \end{matrix} ; q, q \right)$$

$$p_n(x;\mathbf{a}|q^{-1}) = \lambda_n(\mathbf{a}, q)\, {}_8\phi_7 \left( \begin{matrix} q^{-n}, \frac{q^{-n} a_r}{a_p}, \pm q^{1-\frac{n}{2}} \left(\frac{a_r}{a_p}\right)^{\frac{1}{2}}, q^{1-n} a_{rt}, q^{1-n} a_{ru}, \frac{e^{\pm i\theta}}{a_p} \\ \pm q^{-\frac{n}{2}} \left(\frac{a_r}{a_p}\right)^{\frac{1}{2}}, \frac{1}{a_{pt}}, \frac{1}{a_{pu}}, \frac{q a_r}{a_p}, q^{1-n} a_r e^{\pm i\theta} \end{matrix} ; q, \frac{q^n}{a_{tu}} \right)$$

$$p_n(x;\mathbf{a}|q^{-1}) = \frac{\left(-a_{1234} e^{-i\theta}\right)^n}{q^{3\binom{n}{2}}} \frac{(\{\frac{e^{i\theta}}{a_s}\}; q)_n}{(e^{2i\theta}; q)_n} {}_8\phi_7 \left( \begin{matrix} q^{-n}, q^{-n} e^{-2i\theta}, \pm q^{1-\frac{n}{2}} e^{-i\theta}, \{\frac{e^{-i\theta}}{a_s}\} \\ \pm q^{-\frac{n}{2}} e^{-i\theta}, q e^{-2i\theta}, \{q^{1-n} a_s e^{-i\theta}\} \end{matrix} ; q, q^{2-n} a_{1234} \right).$$

# The continuous dual $q$-Hahn polynomials

Define the sets $\mathbf{3} := \{1,2,3\}$, $\mathbf{a} := (a_1,a_2,a_3)$, $q, a_k \in \mathbb{C}^*$, $k \in \mathbf{3}$, and $x = \cos\theta \in [-1,1]$. Let $n \in \mathbb{N}_0$, $p,s,r,t \in \mathbf{3}$, $p,r,t$ distinct and fixed.

$$p_n(x;\mathbf{a}|q) = a_p^{-n}(\{a_pa_s\}_{s\neq p};q)_n\,{}_3\phi_2\left(\begin{matrix} q^{-n}, a_pe^{\pm i\theta} \\ \{a_pa_s\}_{s\neq p}\end{matrix};q,q\right)$$

$$p_n(x;\mathbf{a}|q) = e^{in\theta}\left(a_{pr}, a_te^{-i\theta};q\right)_n\,{}_3\phi_2\left(\begin{matrix} q^{-n}, a_pe^{i\theta}, a_re^{i\theta} \\ a_{pr}, \frac{q^{1-n}e^{i\theta}}{a_t}\end{matrix};q,\frac{qe^{-i\theta}}{a_t}\right)$$

$$p_n(x;\mathbf{a}|q) = q^{-\binom{n}{2}}(-a_p)^{-n}\left(a_pe^{\pm i\theta};q\right)_n\,{}_3\phi_2\left(\begin{matrix} q^{-n}, \{\frac{q^{1-n}}{a_{ps}}\}_{s\neq p} \\ \frac{q^{1-n}e^{\pm i\theta}}{a_p}\end{matrix};q,\frac{q^na_{123}}{a_p}\right)$$

$$p_n(x;\mathbf{a}|q) = e^{in\theta}\left(a_pe^{-i\theta}, a_re^{-i\theta};q\right)_n\,{}_3\phi_2\left(\begin{matrix} q^{-n}, a_te^{i\theta}, \frac{q^{1-n}}{a_{pr}} \\ \frac{q^{1-n}e^{i\theta}}{a_p}, \frac{q^{1-n}e^{i\theta}}{a_r}\end{matrix};q,q\right)$$

$$p_n(x;\mathbf{a}|q) = e^{in\theta}\frac{(\{a_se^{-i\theta}\};q)_n}{(e^{-2i\theta};q)_n}\,{}_8\phi_6\left(\begin{matrix} q^{-n}, q^{-n}e^{2i\theta}, \pm q^{1-\frac{n}{2}}e^{i\theta}, \{a_se^{i\theta}\}, 0 \\ \pm q^{-\frac{n}{2}}e^{i\theta}, qe^{2i\theta}, \{\frac{q^{1-n}e^{i\theta}}{a_s}\}\end{matrix};q,\frac{qe^{-i\theta}}{a_{123}}\right)$$

$$p_n(x;\mathbf{a}|q) = a_p^{-n}\frac{(a_{pt}, a_re^{\pm i\theta};q)_n}{\left(\frac{a_r}{a_p};q\right)_n}\,{}_7\phi_7\left(\begin{matrix} q^{-n}, \frac{q^{-n}a_p}{a_r}, \pm q^{1-\frac{n}{2}}\left(\frac{a_p}{a_r}\right)^{\frac12}, \frac{q^{1-n}}{a_{rt}}, a_pe^{\pm i\theta} \\ \pm q^{-\frac{n}{2}}\left(\frac{a_p}{a_r}\right)^{\frac12}, a_{pt}, \frac{qa_p}{a_r}, \frac{q^{1-n}e^{\pm i\theta}}{a_r}, 0\end{matrix};q,\frac{qa_t}{a_r}\right).$$

# The continuous dual $q$-inverse-Hahn polynomials

$$p_n(x;\mathbf{a}|q^{-1}) = \frac{a_{123}^n}{q^{2\binom{n}{2}}} \left( \left\{ \frac{1}{a_p a_s} \right\}_{s\neq p}; q \right)_n {}_3\phi_2 \left( \begin{matrix} q^{-n}, \frac{e^{\pm i\theta}}{a_p} \\ \{\frac{1}{a_p a_s}\}_{s\neq p} \end{matrix} ; q, \frac{q^n a_p}{a_{123}} \right)$$

$$p_n(x;\mathbf{a}|q^{-1}) = \frac{a_{123}^n}{q^{2\binom{n}{2}}} \left( \frac{1}{a_{pr}}, \frac{e^{i\theta}}{a_t}; q \right)_n {}_3\phi_2 \left( \begin{matrix} q^{-n}, \frac{e^{-i\theta}}{a_p}, \frac{e^{-i\theta}}{a_r} \\ \frac{1}{a_{pr}}, q^{1-n} a_t e^{-i\theta} \end{matrix} ; q, q \right)$$

$$p_n(x;\mathbf{a}|q^{-1}) = \frac{(-a_p)^n}{q^{\binom{n}{2}}} \left( \frac{e^{\pm i\theta}}{a_p}; q \right)_n {}_3\phi_2 \left( \begin{matrix} q^{-n}, \{q^{1-n} a_{ps}\}_{s\neq p} \\ q^{1-n} a_p e^{\pm i\theta} \end{matrix} ; q, q \right)$$

$$p_n(x;\mathbf{a}|q^{-1}) = \frac{\left(-a_{pr} e^{-i\theta}\right)^n}{q^{2\binom{n}{2}}} \left( \frac{e^{i\theta}}{a_p}, \frac{e^{i\theta}}{a_r}; q \right)_n {}_3\phi_2 \left( \begin{matrix} q^{-n}, \frac{e^{-i\theta}}{a_t}, q^{1-n} a_{pr} \\ q^{1-n} a_p e^{-i\theta}, q^{1-n} a_r e^{-i\theta} \end{matrix} ; q, q a_t e^{-i\theta} \right)$$

$$p_n(x;\mathbf{a}|q^{-1}) = \frac{(a_{pt}, a_r e^{\pm i\theta}; q)_n}{a_p^n \left(\frac{a_r}{a_p}; q\right)_n} {}_7\phi_6 \left( \begin{matrix} q^{-n}, \frac{q^{-n} a_r}{a_p}, \pm q^{1-\frac{n}{2}} \left(\frac{a_r}{a_p}\right)^{\frac{1}{2}}, \frac{q^{1-n}}{a_{pt}}, a_r e^{\pm i\theta} \\ \pm q^{-\frac{n}{2}} \left(\frac{a_r}{a_p}\right)^{\frac{1}{2}}, a_{rt}, \frac{q a_r}{a_p}, \frac{q^{1-n} e^{\pm i\theta}}{a_p} \end{matrix} ; q, \frac{q^{2-n} a_t}{a_{pr}} \right)$$

$$p_n(x;\mathbf{a}|q^{-1}) = \lambda_n(\mathbf{a}, q)\, {}_7\phi_7 \left( \begin{matrix} q^{-n}, \pm q^{1-\frac{n}{2}} e^{-i\theta}, q^{-n} e^{-2i\theta}, \left\{ \frac{e^{-i\theta}}{a_s} \right\} \\ \pm q^{-\frac{n}{2}} e^{-i\theta}, q e^{-2i\theta}, \{q^{1-n} a_s e^{-i\theta}\}, 0 \end{matrix} ; q, q^{2-n} a_{123} e^{-i\theta} \right).$$

# The Al-Salam-Chihara polynomials

Define the sets $\mathbf{2} := \{1,2\}$, $\mathbf{a} := (a_1, a_2)$, $q, a_k \in \mathbb{C}^*$, $k \in \mathbf{2}$, and $x = \cos\theta \in [-1,1]$. Let $n \in \mathbb{N}_0$, $p, r, s \in \mathbf{2}$, $p, r$ distinct and fixed.

$$Q_n(x;\mathbf{a}|q) = a_p^{-n}(a_{12};q)_n\, {}_3\phi_2\left(\begin{matrix} q^{-n}, a_p e^{\pm i\theta} \\ a_{12}, 0 \end{matrix}; q, q\right)$$

$$Q_n(x;\mathbf{a}|q) = q^{-\binom{n}{2}}(-a_p)^{-n}(a_p e^{\pm i\theta};q)_n\, {}_2\phi_2\left(\begin{matrix} q^{-n}, \frac{q^{1-n}}{a_{12}} \\ \frac{q^{1-n}e^{\pm i\theta}}{a_p} \end{matrix}; q, \frac{q a_{12}}{a_p^2}\right)$$

$$Q_n(x;\mathbf{a}|q) = e^{in\theta}(a_{12};q)_n\, {}_3\phi_1\left(\begin{matrix} q^{-n}, \{a_s e^{i\theta}\} \\ a_{12} \end{matrix}; q, q^n e^{-2i\theta}\right)$$

$$Q_n(x;\mathbf{a}|q) = e^{-in\theta}(a_p e^{i\theta};q)_n\, {}_2\phi_1\left(\begin{matrix} q^{-n}, a_r e^{-i\theta} \\ \frac{q^{1-n}e^{-i\theta}}{a_p} \end{matrix}; q, \frac{q e^{i\theta}}{a_p}\right)$$

$$Q_n(x;\mathbf{a}|q) = e^{-in\theta}(\{a_s e^{i\theta}\};q)_n\, {}_3\phi_2\left(\begin{matrix} q^{-n}, \frac{q^{1-n}}{a_{12}}, 0 \\ \left\{\frac{q^{1-n}e^{-i\theta}}{a_s}\right\} \end{matrix}; q, q\right)$$

$$Q_n(x;\mathbf{a}|q) = a_p^{-n}\left(\{a_p a_s\}_{s\neq p};q\right)_n \,{}_6\phi_7\left(\begin{matrix} q^{-n}, \frac{q^{-n}a_p}{a_r}, \pm q^{1-\frac{n}{2}}\left(\frac{a_p}{a_r}\right)^{\frac{1}{2}}, a_p e^{\pm i\theta} \\ \pm q^{-\frac{n}{2}}\left(\frac{a_p}{a_r}\right)^{\frac{1}{2}}, \frac{qa_p}{a_r}, \frac{q^{1-\frac{n}{2}}e^{\pm i\theta}}{a_r}, 0, 0 \end{matrix}; q, \frac{q^{2-n}}{a_r^2}\right)$$

$$Q_n(x;\mathbf{a}|q) = e^{in\theta}\frac{\left(\{a_s e^{-i\theta}\};q\right)_n}{(e^{-2i\theta};q)_n}\,{}_8\phi_5\left(\begin{matrix} q^{-n}, \pm q^{1-\frac{n}{2}}e^{i\theta}, q^{-n}e^{2i\theta}, \{a_s e^{i\theta}\}, 0, 0 \\ \pm q^{-\frac{n}{2}}e^{i\theta}, qe^{2i\theta}, \left\{\frac{q^{1-n}e^{i\theta}}{a_s}\right\} \end{matrix}; q, \frac{q^n e^{-2i\theta}}{a_{12}}\right)$$

Equivalent results can be obtained in all these cases.

Let $n \in \mathbb{N}_0$, $a, b, c, d, q \in \mathbb{C}^*$, such that $|q| \neq 1$. Then, one has the following transformation formula for a terminating ${}_3\phi_1$:

$$ {}_3\phi_1\left(\begin{matrix} q^{-n}, c, d \\ q^{-n}\frac{cd}{b} \end{matrix}; q, \frac{1}{b}\right) = \frac{1}{(q^n b)^n}\, {}_3\phi_1\left(\begin{matrix} q^{-n}, \frac{q^{-n}c}{b}, \frac{q^{-n}d}{b} \\ q^{-n}\frac{cd}{b} \end{matrix}; q, q^{2n}b\right). $$

Also, the following transformation formula for a terminating ${}_8\phi_5$ holds:

$$ {}_8\phi_5\left(\begin{matrix} q^{-n}, \pm q\sqrt{b}, b, c, d, 0, 0 \\ \pm\sqrt{b}, q^{n+1}b, \frac{qb}{c}, \frac{qb}{d} \end{matrix}; q, \frac{q^n}{cd}\right) = \frac{\left(c, d, \frac{q^{-n}}{b}; q\right)_n}{\left(\frac{cq^{-n}}{b}, \frac{q^{-n}d}{b}, q^n b; q\right)_n} $$

$$ \times \frac{1}{(q^n b)^n}\, {}_8\phi_5\left(\begin{matrix} q^{-n}, \pm\frac{q^{1-n}}{\sqrt{b}}, \frac{q^{-2n}}{b}, \frac{q^{-n}c}{b}, \frac{q^{-n}d}{b}, 0, 0 \\ \pm\frac{q^{-n}}{\sqrt{b}}, \frac{q^{1-n}}{b}, \frac{q^{1-n}}{c}, \frac{q^{1-n}}{d} \end{matrix}; q, \frac{q^{3n}b^2}{cd}\right). $$

# The continuous big $q$-Hermite polynomials

Let $n \in \mathbb{N}_0$, $a, q \in \mathbb{C}^*$, $|q| \neq 1$.

$$H_n(x;a|q) = \frac{1}{a^n}\, {}_3\phi_2\left(\begin{matrix} q^{-n}, ae^{\pm i\theta} \\ 0, 0 \end{matrix}; q, q\right)$$

$$H_n(x;a|q) = \frac{q^{-\binom{n}{2}}}{(-a)^n}(ae^{\pm i\theta};q)_n\, {}_1\phi_2\left(\begin{matrix} q^{-n} \\ \frac{q^{1-n}e^{\pm i\theta}}{a} \end{matrix}; q, \frac{q^{2-n}}{a^2}\right)$$

$$H_n(x;a|q) = e^{-in\theta}(ae^{i\theta};q)_n\, {}_2\phi_1\left(\begin{matrix} q^{-n}, 0 \\ \frac{q^{1-n}e^{-i\theta}}{a} \end{matrix}; q, \frac{qe^{i\theta}}{a}\right)$$

$$H_n(x;a|q) = e^{in\theta}\, {}_2\phi_0\left(\begin{matrix} q^{-n}, ae^{i\theta} \\ - \end{matrix}; q, q^n e^{-2i\theta}\right)$$

$$H_n(x;a|q) = e^{in\theta}\frac{(ae^{-i\theta};q)_n}{(e^{-2i\theta};q)_n}\, {}_8\phi_4\left(\begin{matrix} q^{-n}, q^{-n}e^{2i\theta}, \pm q^{1-\frac{n}{2}}e^{i\theta}, ae^{i\theta}, 0, 0, 0 \\ \pm q^{-\frac{n}{2}}e^{i\theta}, qe^{2i\theta}, \frac{q^{1-n}e^{i\theta}}{a} \end{matrix}; q, \frac{q^{2n-1}e^{-3i\theta}}{a}\right).$$

# Further reading


- Koekoek, R.; Lesky, P. A.; Swarttouw, R. F. *Hypergeometric orthogonal polynomials and their q-analogues*; Springer Monographs in Mathematics, Springer-Verlag: Berlin, 2010; pp. xx+578. With a foreword by Tom H. Koornwinder.
- Koornwinder, T. H. Additions to the formula lists in "Hypergeometric orthogonal polynomials and their $q$-analogues" by Koekoek, Lesky and Swarttouw. arXiv:1401.0815v2 **2015**.
- Gasper, G.; Rahman, M. *Basic hypergeometric series*, second ed.; Vol. 96, Encyclopedia of Mathematics and its Applications, Cambridge University Press: Cambridge, 2004; p. xxvi+428. With a foreword by Richard Askey.

# That's all Falks ;)

Thank you for your attention