Utility of integral representations for basic hypergeometric functions and OP

Short visit Universidade da Beira Interior

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Outline

Purpose: Demonstrate the utility of integral representations for problems in basic hypergeometric functions and orthogonal polynomials.

Applications: Integral representations, generating functions, transformation formulas, and solutions to outstanding problems.

- Integral representations simplify complex summations.
- Strong connections to the q-Askey scheme.
- Historical development by Bailey, Slater, Askey, Roy, Gasper, Rahman, etc.

Preliminaries

- We start with really basic details

$$\pm a := \{a, -a\}.$$

We also adopt an analogous notation

$$e^{\pm i\theta} := \{e^{i\theta}, e^{-i\theta}\}.$$

- A q-analog of the raising factorial (q-Pochhmmer symbol):

$$(a;q)_n := (1-a)(1-aq)\cdots(1-aq^{n-1}).$$

- Let b a complex number:

$$(a;q)_b := \frac{(a;q)_\infty}{(aq^b;q)_\infty},$$

and

$$(a_1,...,a_k;q)_b := (a_1;q)_b \cdots (a_k;q)_b,$$

- The theta function of nome q

$$\vartheta(x;q) := (x, q/x; q)_\infty,$$

- Basic hypergeometric series ${}_r\phi_s$ is defined as

$$ {}_{r+1}\phi_s\left(\begin{matrix} a_1, \ldots, a_{r+1} \\ b_1, \ldots, b_s \end{matrix}; q, z\right) := \sum_{k=0}^{\infty} \frac{(a_1, \ldots, a_{r+1}; q)_k}{(q, b_1, \ldots, b_s; q)_k} \left((-1)^k q^{\binom{k}{2}}\right)^{s-r} z^k. $$

For $s > r$, ${}_{r+1}\phi_s$ is an entire function of z, for $s = r$ then ${}_{r+1}\phi_s$ is convergent for $|z| < 1$, and for $s < r$ the series is divergent.

- Special notation by Bult & Rains.

$$ {}_{r+1}\phi_s^{-p}\left(\begin{matrix} a_1, \ldots, a_{r+1} \\ b_1, \ldots, b_s \end{matrix}; q, z\right) := {}_{r+p+1}\phi_s\left(\begin{matrix} a_1, a_2, \ldots, a_{r+1}, \overbrace{0, \ldots, 0}^{p} \\ b_1, b_2, \ldots, b_s \end{matrix}; z\right), $$

- Define the Jackson q-integral as

$$ \int_a^b f(u; q)\mathrm{d}_q u = (1-q)b\sum_{n=0}^{\infty} q^n f(q^n b; q) - (1-q)a\sum_{n=0}^{\infty} q^n f(q^n a; q). $$

The Askey-Wilson polynomials

The Askey–Wilson polynomials can be defined in terms of the terminating basic hypergeometric series

$$p_n(x;a,b,c,d|q) := a^{-n}(ab,ac,ad;q)_n\, {}_4\phi_3\left(\begin{matrix} q^{-n}; q^{n-1}abcd, ae^{\pm i\theta} \\ ab, ac, ad \end{matrix}; q, q\right)$$

where $x = \cos\theta$.
The Askey–Wilson polynomials are orthogonal on $(-1,1)$ with respect to the weight function

$$w_q(\cos\theta;\mathbf{a}) := \frac{(e^{\pm 2i\theta};q)_\infty}{(\mathbf{a}e^{\pm i\theta};q)_\infty} = \frac{(\pm e^{\pm i\theta}, \pm q^{\frac{1}{2}}e^{\pm i\theta};q)_\infty}{(\mathbf{a}e^{\pm i\theta};q)_\infty},$$

where $\mathbf{a} := \{a,b,c,d\}$.

$$\int_0^\pi p_m(x;\mathbf{a}|q)p_n(x;\mathbf{a}|q)w_q(x;\mathbf{a})\,\mathrm{d}\theta = h_n(\mathbf{a};q)\delta_{m,n},$$

where

$$h_n(\mathbf{a};q) := \frac{2\pi(q^{n-1}a_{1234};q)_n(q^{2n}a_{1234};q)_\infty}{(q^{n+1},q^na_{12},q^na_{13},q^na_{14},q^na_{23},q^na_{24},q^na_{34};q)_\infty},$$

and $a_{ij} := a_ia_j$, $a_{ijk} := a_ia_ja_k$, ...

Examples

Generalized q-beta integrals. The Askey–Wilson and the Nassrallah–Rahman integrals

The Askey–Wilson integral.
Let $\mathbf{a} := \{a_1, a_2, a_3, a_4\}$. Then

$$\int_0^\pi \frac{(e^{\pm 2i\theta};q)_\infty}{(\mathbf{a}e^{\pm i\theta};q)_\infty}\,\mathrm{d}\theta = \frac{2\pi(a_{1234};q)_\infty}{(q,a_{12},a_{13},a_{14},a_{23},a_{24},a_{34};q)_\infty}.$$

The Nassrallah–Rahman integral.
Let $\mathbf{a} := \{a_1, a_2, a_3, a_4, a_5\}$. Then

$$\int_0^\pi \frac{(e^{\pm 2i\theta},\lambda e^{\pm i\theta};q)_\infty}{(\mathbf{a}e^{\pm i\theta};q)_\infty}\mathrm{d}\theta = \frac{2\pi(\lambda\mathbf{a},\lambda^{-1}a_{12345};q)_\infty}{(q,a_{12},\dots,a_{45},\lambda^2;q)_\infty}\,{}_8W_7\left(\frac{\lambda^2}{q};\frac{\lambda}{\mathbf{a}};q,\frac{a_{12345}}{\lambda}\right)$$

where

$${}_{r+1}W_r(b;a_4,\dots,a_{r+1};q,z) := {}_{r+1}\phi_r\left(\begin{matrix}\pm q\sqrt{b},b,a_4,\dots,a_{r+1}\\ \pm\sqrt{b},\frac{qb}{a_4},\dots,\frac{qb}{a_{r+1}}\end{matrix};q,z\right).$$

The Rahman integral generalizes the Nassrallah–Rahman integral.

Let $\mathbf{a} := \{a_1, a_2, a_3, a_4, a_5, a_6\}$. Then

$$\int_0^\pi \frac{(e^{\pm 2i\theta}, \lambda e^{\pm i\theta}, \mu e^{\pm i\theta}; q)_\infty}{(\mathbf{a} e^{\pm i\theta}; q)_\infty} \mathrm{d}\theta = \frac{2\pi}{(q, a_{12}, \ldots, a_{56}; q)_\infty}$$
$$\times \left(\frac{(\lambda \mathbf{a}, \frac{\mu}{\mathbf{a}}; q)_\infty}{(\lambda^2, \mu/\lambda; q)_\infty} {}_{10}W_9\left(\frac{\lambda^2}{q}; \frac{\lambda\mu}{q}, \frac{\lambda}{\mathbf{a}}; q, q\right)\right.$$
$$\left. + \frac{(\mu \mathbf{a}, \frac{\lambda}{\mathbf{a}}; q)_\infty}{(\mu^2, \lambda/\mu; q)_\infty} {}_{10}W_9\left(\frac{\mu^2}{q}; \frac{\lambda\mu}{q}, \frac{\mu}{\mathbf{a}}; q, q\right)\right),$$

Main result

Main result

Let $q \in \mathbb{C}$, $m \in \mathbb{Z}$, $t \in \mathbb{C}^*$, $\sigma \in (0, \infty)$, $\mathbf{a} := \{a_1, \ldots, a_A\}$, $\mathbf{b} := \{b_1, \ldots, b_B\}$, $\mathbf{c} := \{c_1, \ldots, c_C\}$, $\mathbf{d} := \{d_1, \ldots, d_D\}$ be sets of complex numbers with cardinality $A, B, C, D \in \mathbb{N}_0$ (not all zero) respectively with $|c_k| < \sigma/|t|$, $|d_l| < 1/\sigma$, for any $a_i, b_j, c_k, d_l \in \mathbb{C}$ elements of $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$, and $z = e^{i\psi}$. Define

$$G_{m,t} := \frac{(q;q)_\infty}{2\pi} \left(\frac{\sqrt{t}}{\sigma}\right)^m \int_{-\pi}^{\pi} \frac{(\mathbf{b}\frac{\sigma}{z}, t\mathbf{a}\frac{z}{\sigma}; q)_\infty}{(\mathbf{d}\frac{\sigma}{z}, t\mathbf{c}\frac{z}{\sigma}; q)_\infty} e^{im\psi} \mathrm{d}\psi,$$

such that the integral exists. Then

$$G_{m,t}(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}; \sigma, q) = G_{-m,t}(\mathbf{b}, \mathbf{a}, \mathbf{d}, \mathbf{c}; \sigma, q),$$

if $|c_k|, |d_l| < \min\{1/\sigma, \sigma/|t|\}$. Furthermore, let $td_lc_k \notin \Omega_q$. If $D \geq B$, $d_l/d_{l'} \notin \Omega_q$, $l \neq l'$, then

$$G_{m,t} = t^{\frac{m}{2}} \sum_{k=1}^{D} \frac{(td_k\mathbf{a}, \mathbf{b}/d_k; q)_\infty d_k^m}{(td_k\mathbf{c}, \mathbf{d}_{[k]}/d_k; q)_\infty} {}_{B+C}\phi_{A+D-1}^{C-A}\left(\begin{matrix} td_k\mathbf{c}, qd_k/\mathbf{b} \\ td_k\mathbf{a}, qd_k/\mathbf{d}_{[k]} \end{matrix}; q, q^m(qd_k)^{D-B}\frac{b_1\cdots b_B}{d_1\cdots d_D}\right).$$

A CONSEQUENCE

Let $\mathbf{a}$, $\mathbf{c}$, $\mathbf{d}$, q, s, f, σ, $z = e^{i\psi}$ as in the former result. Then

$$\int_{-\pi}^{\pi} \frac{((fd_1, \frac{q}{f}d_2)\frac{\sigma}{z}, (\frac{f}{d_2}, \frac{q}{fd_1}, \mathbf{a})\frac{z}{\sigma}; q)_\infty}{((d_1, d_2)\frac{\sigma}{z}, \mathbf{c}\frac{z}{\sigma}; q)_\infty} \mathrm{d}\psi = \frac{2\pi\sqrt{\frac{d_2}{d_1}}\vartheta(f, f\frac{d_1}{d_2}; q)}{(1-q)s(q,q;q)_\infty \vartheta(\frac{d_2}{d_1}; q)}$$

$$\times \int_{s\sqrt{\frac{d_2}{d_1}}}^{s\sqrt{\frac{d_1}{d_2}}} \frac{((q\sqrt{d_1/d_2}, q\sqrt{d_2/d_1}, \mathbf{a}\sqrt{d_1d_2})\frac{u}{s}; q)_\infty}{(\mathbf{c}\sqrt{d_1d_2}\frac{u}{s}; q)_\infty} \mathrm{d}_q u.$$

Applications

$$ {}_2\phi_1\left(\begin{matrix} a, b \\ c \end{matrix}; q, z\right) = -\frac{1}{2i}\frac{(a,b;q)_\infty}{(q,c;q)_\infty}\int_{-i\infty}^{i\infty}\frac{((q,c)q^s;q)_\infty}{((a,b)q^s;q)_\infty}\frac{(-z)^s}{\sin(\pi s)}\mathrm{d}s, $$

where $\pm i\infty := \pm\lim_{x\uparrow\infty} ix$, where $x\in(0,\infty)$.

Result: Let $a,b,c,z,q\in\mathbb{C}^*$, with $|z|<1$, $\tau\in(0,1)$, $w=e^{i\eta}$. Then

$$ {}_2\phi_1\left(\begin{matrix} a, b \\ c \end{matrix}; q, z\right) = \frac{(q,a,\frac{c}{b},\frac{abz}{c};q)_\infty}{2\pi\vartheta(f,f\frac{c}{bz};q)}\int_{-\pi}^{\pi}\frac{((f\sqrt{\frac{c}{bz}},\frac{q}{f}\sqrt{\frac{bz}{c}})\frac{\tau}{w},(f\sqrt{\frac{c}{bz}},\frac{q}{f}\sqrt{\frac{bz}{c}},\sqrt{bcz})\frac{w}{\tau};q)_\infty}{((\sqrt{\frac{c}{bz}},\sqrt{\frac{bz}{c}})\frac{\tau}{w},(\sqrt{\frac{bz}{c}}a,\sqrt{\frac{cz}{b}})\frac{w}{\tau};q)_\infty}\mathrm{d}\eta $$

$$ {}_2\phi_1\left(\begin{matrix} a, b \\ c \end{matrix}; q, z\right) = \frac{(q,a,b,\frac{c}{a},\frac{c}{b},\frac{abz}{c};q)_\infty}{2\pi\vartheta(f,f\frac{ab}{c};q)(c;q)_\infty}\int_{-\pi}^{\pi}\frac{((f\sqrt{\frac{ab}{c}},\frac{q}{f}\sqrt{\frac{c}{ab}})\frac{\tau}{w},(f\sqrt{\frac{ab}{c}},\frac{q}{f}\sqrt{\frac{c}{ab}})\frac{w}{\tau};q)_\infty}{((\sqrt{\frac{ab}{c}},\sqrt{\frac{c}{ab}})\frac{\tau}{w},(\sqrt{\frac{ac}{b}},\sqrt{\frac{bc}{a}},\sqrt{\frac{ab}{c}}z)\frac{w}{\tau};q)_\infty}\mathrm{d}\eta. $$

The expression

$$\sum_{k=1}^{D} \frac{(td_k\,\mathbf{a}, d_k^{-1}\,\mathbf{b}; q)_\infty}{(td_k\mathbf{c}, d_k^{-1}\,\mathbf{d}_{[k]}; q)_\infty}\ {}_{B+C}\phi^{A-C}_{A+D-1}\left(\begin{matrix} td_k\,\mathbf{c}, qd_k\,\mathbf{b}^{-1} \\ td_k\,\mathbf{a}, qd_k\mathbf{d}_{[k]}^{-1} \end{matrix}; q, \frac{b_1\cdots b_B}{d_1\cdots d_D}\right)$$

is equal to

$$\sum_{k=1}^{C} \frac{(tb_k\,\mathbf{c}, b_k^{-1}\,\mathbf{a}; q)_\infty}{(tc_k\mathbf{d}, c_k^{-1}\,\mathbf{c}_{[k]}; q)_\infty}\ {}_{A+D}\phi^{B-D}_{B+C-1}\left(\begin{matrix} tc_k\,\mathbf{d}, qc_k\,\mathbf{a}^{-1} \\ tc_k\,\mathbf{b}, qc_k\mathbf{c}_{[k]}^{-1} \end{matrix}; q, \frac{a_1\cdots a_A}{c_1\cdots c_C}\right).$$

The Askey–Wilson polynomials can be written as follows:

$$
{}_4\phi_3\left(\begin{matrix} q^{-n}, q^{n-1}abcd, ae^{\pm i\theta} \\ ab, ac, ad \end{matrix}; q, q\right) = \frac{(a^2cd, cd; q)_n(\frac{qa}{b}, \frac{q}{ab}, acd\, e^{\pm i\theta}; q)_\infty}{(acd\, e^{\pm i\theta}; q)_n(\frac{q}{b}e^{\pm i\theta}, a^2cd, cd; q)_\infty}
$$

$$
\times\ {}_8W_7\left(q^{n-1}a^2cd; q^n ac, q^n ad, q^{n-1}abcd, ae^{\pm i\theta}; q, \frac{q^{1-n}}{ab}\right),
$$

where $x = \cos\theta$, $|q^{1-n}| < |ab|$.

Let $k, p \in \{1,2,3,4\}$, $\mathbf{a} := \{a_1, a_2, a_3, a_4\}$, $t, q, a_k \in \mathbb{C}^*$, $x = \cos\theta \in [-1,1]$, $|a_p t| < 1$. Then

$$\sum_{n=0}^{\infty} \frac{t^n\,(q^{-1}a_{1234};q)_n\,p_n(x;\mathbf{a}|q)}{(q,\{a_pa_s\}_{s\neq p};q)_n} = \frac{(ta_{1234}(qa_p)^{-1};q)_\infty}{(ta_p^{-1};q)_\infty}\,{}_6\phi_5\left(\begin{matrix}\pm(q^{-1}a_{1234})^{\frac12}, \pm(a_{1234})^{\frac12}, a_pe^{\pm i\theta} \\ \{a_pa_s\}_{s\neq p}, ta_{1234}(qa_p)^{-1}, qa_pt^{-1}\end{matrix}; q, q\right)$$

$$+ \frac{(\{ta_s\}_{s\neq p}, q^{-1}a_{1234}, a_pe^{\pm i\theta};q)_\infty}{(\{a_pa_s\}_{s\neq p}, a_pt^{-1}, te^{\pm i\theta};q)_\infty}\,{}_6\phi_5\left(\begin{matrix}\pm ta_p^{-1}(q^{-1}a_{1234})^{\frac12}, \pm ta_p^{-1}(a_{1234})^{\frac12}, te^{\pm i\theta} \\ \{ta_s\}_{s\neq p}, q^{-1}a_{1234}(ta_p^{-1})^2, qta_p^{-1}\end{matrix}; q, q\right).$$

Thank you!

Slides: www.rscosan.com/talk/seminar29

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